Algebraic Topology: On Some Results Of Injective for Topological Modules

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Abstract:

The main objective of this research is to study some properties of Injective for topological module. More result in this paper are obtained we show you can find Injective topological module.

Keywords: Topological module, Topological submodule, the tensor product of Topological submodule, injective Topological module.

بعض النتائج حول المقاسات التبولوجيه الاغماريه

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الملخص:

في هذا البحث تم دراسة بعض الخصائص المقاسية الجبرية تبولوجيا" وخصوصا" المقاسات الجبرية الاغماريه وخرجنا ببعض النتائج كما موضحه في البحث، اذ بينا كيفيه إيجاد المقاسات التبولوجيه الاغماريه من مقاسات أخرى.

الكلمات المفتاحية: المقاسات التبولوجيه، المقاسات الجزئية التبولوجيه، الضرب التنسوري للمقاسات الجزئية التبولوجيه، المقاسات التبولوجيه الاغمارية.

Article History:

Algebraic topology is one of the important branches of mathematics and embodies the relationships between the algebra and the topology. The true beginning of the study of algebraic topology in 1920s through the study of topological group. In 1955, Cabaske introduced a definition of topological module, and the Partial topological measurement, the number of research like Dikran Dikran. In this paper we study topological modules especially injective topological modules. The Important of this work is to evaluate the tensor product of injective topological module.

1-Introduction

In this paper, we have the principal goal is to study a topology property of important algebraic construction namely the injective module. We use a new tool with a injective module which is a tensor product of modules. Therefore all topogy submodules in this notion are a tensor product. The meaning of the tensor module introduced in this notion and the important fact of this article is to explain the injective module when all submodules are tensor. Finally, several results have been obtained about the injective topology module.

The scientists needed to consider the topography module. In 1955, Cabaske presented the meaning of the topogical module. A definition and a few properties of the topogical module and topogical submodule can discover it in [1,7]. To contemplate the remainder module we need to present a simple meaning of shape similarity topogical module:

Let $M: D \to D_1$ be a mapping between two topogical modules. μ is called a shape similarity topogical module if H is a shape similarity and continuous [4], and in [1]

A tensor module of topography R-modules H and S have been defined in and more details about a tensor concept in [4]. A tensor product of topography R-module H, S is a topography R-module dented by $H \otimes S$ together with R-bilinear mapping $T: H \times S \to H \otimes S$ such that for every R-bilinear mapping $\psi: H \times S \to X$. There exists a unique linear mapping $\emptyset: H \otimes S \to X$ such that the:

Commutes that is $\psi = \emptyset \ o \ T$.[2]

Leave D alone a left topography R-module. A subset H of D is called a topogical submodule if**:**

1. H is a submodule of D
2. H is a topogical subspace of topography space D.

In [5,6] Let D, D' be a topogical module. The planning v: D→D' is called homomorphism topogical module if

1. v is a shape similarity module.
2. v is a continuous map

And v is called homeomorphism topology module if

- v is isomorphism module
- v is shape similarity topology.

Now the mapping $v: D \to D'$ is called regular embedding if it's the embedding mapping and $v(D)$ is open of D' where D, D' be a topological module.

Let D be a topologic module and $H_1 \otimes H_2$ is a topological submodule of E then E is called exending injective for $H_1 \otimes H_2$ is a D is injective and $H_1 \otimes H_2$ is open of D.

Let $D_1 \otimes D_2$ be a topological module on topological ring. $S_1 \otimes S_2$ is called maximal essential extension for $D_1 \otimes D_2$ if:

1) $S_1 \otimes S_2$ is essential extension for $D_1 \otimes D_2$
2) If $\acute{S}_1 \otimes S'_2$ is proper $S_1 \otimes S_2$ is essential for $D_1 \otimes D_2$

Then $\acute{S}_1 \otimes \acute{S}_2$ is not essential extension for $D_1 \otimes D_2$.

A topological module $D_1 \otimes D_2$ on topological ring R is essential for topological submodule $M_1 \otimes M_2$ then $D_1 \otimes D_2$ is minimal essential injective for $M_1 \otimes M_2$ if :

1) $D_1 \otimes D_2$ is injective.
2) If $\acute{D}_1 \otimes D'_2$ is topological submodule of $D_1 \otimes D_2$ contain proper $M_1 \otimes M_2$.

Then $D_1 \otimes D_2$is not injective.

Let D be topological module and let $M_1 \otimes M_2$ be a topological submodule of D then D is called essential extension of $M_1 \otimes M_2$ if $M_1 \otimes M_2$ is open in D and

$(S_1 \otimes S_2) \cap (M_1 \otimes M_2) \neq 0$[3,2]

Where $S_1 \otimes S_2$ is any topological submodule of D.

2. on some Results of Injective Topological module

In this section we give some results about the work. But before that we need to introduce the following definition.

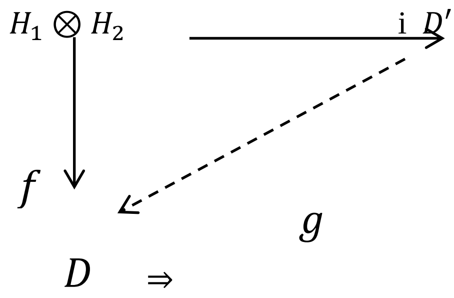
Definition (2-1)

Let D be a left topological module. Then D is called injective if it has the properties $\acute{H}_1 \otimes \acute{H}_2$ is a left is left open topological submodule of topological module $H_1 \otimes H_2$ and $v: \acute{H}_1 \otimes \acute{H}_2 \to D$ be a homomorphism topological module then v extendent to homomorphism topological module $H_1 \otimes H_2$ to D.

Proposition (2.2):

Let D be a discrete of topological module $D, then\ D$ is injective if and only if for any closed of topological submodule $S_1 \otimes S_2$ of a topological module $\grave{D}'$. homomorphism of topological module
$f: S_1 \otimes S_2 \to D$ extended to D'.

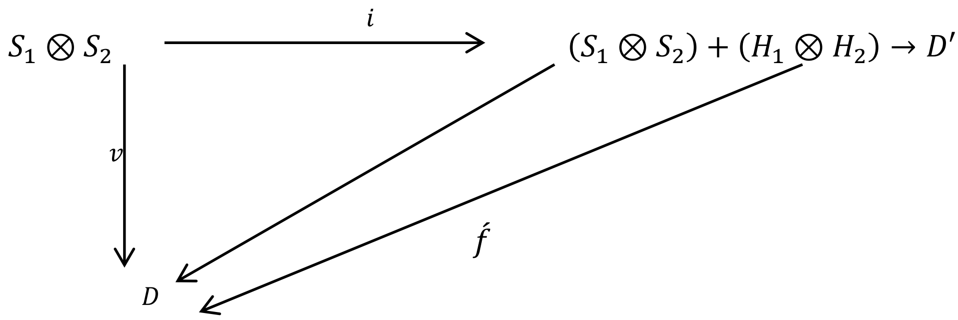
Proof: Let $S_1 \otimes S_2$ be left open of topological submodule of topological module D'. hence its closed and by assumption we get extended $g: D' \to D$ and by definition(2-1) D is injective.



Let v be a homomorphism of topological module: $S_1 \otimes S_2 \to D$, $ker(v)$ is open of topological submodule of $\grave{D}$, and therefore contain intersection

$(S_1 \otimes S_2) \cap (H_1 \otimes H_2)$. of $\grave{D}$. we can extension v to $(S_1 \otimes S_2) + (M_1 \otimes M_2)$

$v\big((s_1 \otimes s_2) + (h_1 \otimes h_2)\big) = v(s_1 \otimes s_2)$



Since D is injective of topological module and $(S_1 \otimes S_2) + (H_1 \otimes H_2)$ is open topological submodule of $\acute{D}$ then v is extension to homomorphism of topological module from D' to D, then v extension to $\acute{D}$.

Proposition (2.3)

Let D be a topological module on topological ring R, D is injective if and only if for every injective regular of any topological module $Z_1 \otimes Z_2$ to any topological module $Q_1 \otimes Q_2$ and any homomorphism of topological module

$g: Z_1 \otimes Z_2 \to D$ there exists homomorphism topological module

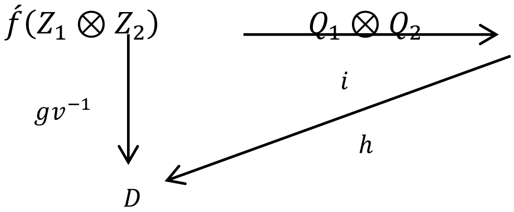
$h: Q_1 \otimes Q_2 \to D$ such that $g = h \circ v$.

proof:

suppose that D is injective topological module. Since
$f: Z_1 \otimes Z_2 \to Q_1 \otimes Q_2$ is injective regular map, then $f(Z_1 \otimes Z_2)$ is open topological submodule of $Q_1 \otimes Q_2$

$gv^{-1}: f(Z_1 \otimes Z_2) \to D$ extension to homomorphism of topological module

$h: Q_1 \otimes Q_2 \to D$ it mean that $h \circ i = gv^{-1}$.



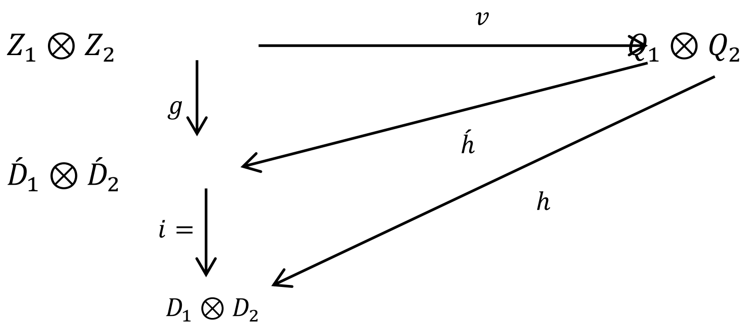
And then $g = h \circ v \Rightarrow$

Let $\acute{H}_1 \otimes \acute{H}_2$ be open topological submodule of topological module D' and let $g: H'_1 \otimes \acute{H}_2 \to D$ be a homomorphism topological, i:. $\acute{H}_1 \otimes \acute{H}_2 \to H_1 \otimes H_2$,since $\acute{H}_1 \otimes \acute{H}_2$ be open topological submodule of $H_1 \otimes H_2$ then i is regular injective map.and then there exists homomorphism topological module h: $H_1 \otimes H_2 \to D$ such that g=h ∘ i hence h is extended to g.

Proposition (2.4):

Let $D_1 \otimes D_2$ be a left injective topological module on the topological ring R and let $\acute{D}_1 \otimes \acute{D}_2$ be a left topological module of the same topological ring if $I: \acute{D}_1 \otimes \acute{D}_2 \to D_1 \otimes D_2$ be homomorphism of topological ring then $\acute{D}_1 \otimes \acute{D}_2$ is injective.

Proof: By the diagram



Such that $Z_1 \otimes Z_2$, $Q_1 \otimes Q_2$ are two topological R-module, and $v: Z_1 \otimes Z_2 \to Q_1 \otimes Q_1$. Since $D_1 \otimes D_2$ is injective topological module then there exists homomorphism of topological module $h: Q_1 \otimes Q_1 \to D_1 \otimes D_2$ such that

$h \circ v = i \circ g.$

Now define $\acute{h}: Q_1 \otimes Q_2 \to \acute{D}_1 \otimes D'_2$ such that

$\acute{h}(q_1 \otimes q_2) = (i^{-1} \circ h)(q_1 \otimes q_2)$ which can be confirmed that is homeomorphism of topological module for all $a_1 \otimes a_2 \in Z_1 \otimes Z_2$

$$(\acute{h} \circ v)(a_1 \otimes a_2) = (i^{-1} \circ h)(a_1 \otimes a_2)$$

$$= (i^{-1} \circ h \circ g)(a_1 \otimes a_2)$$

$$= g(a_1 \otimes a_2)$$

Hence $D'_1 \otimes \acute{D}_2$ is injective.

Proposition (2.5)
Let $D_1, D_2 \ldots D_{n-1}, D_n$ be a left topological modules, and let $D_1, D_2 \ldots D_{n-1}$ be a left injective topological module on the topological ring R if $i: D_n \to D_{n-1}$ homomorphism of topological ring then D_n is injective.
Proof:
Let Z and Q are topological module on the same topological ring $R, f: Z \to Q$ and let $g: Z \to D_n$ be a homomorphism of topological module since D_{n-1} is injective topological module then there exists homomorphism of topological module $h: Q \to D_{n-1}$ such that $h \circ v = i \circ g$.
Now define $h_n: Q \to D_n$ $h_n(b) = \{i^{-1} \circ h(b)\}$

$(h_n \circ v)(b) = \{i^{-1} \circ h \circ v(b)\}$

$= (i^{-1} \circ g \circ i)(b)$

$= g(b)$

If $b = 0$

$h(0) = \left\{i \circ h(0) = \{0\}\right\}$

hence h is homeomorphism of topological module

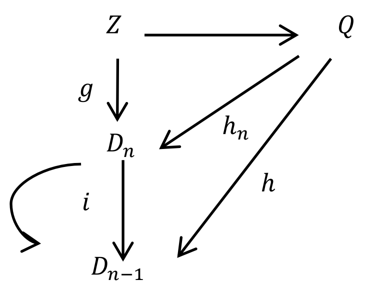
for $a \in Z$

$(h \circ v)(a) = (i^{-1} \circ h \circ v)(a)$

v

$= (i^{-1} \circ i \circ g)(a)$

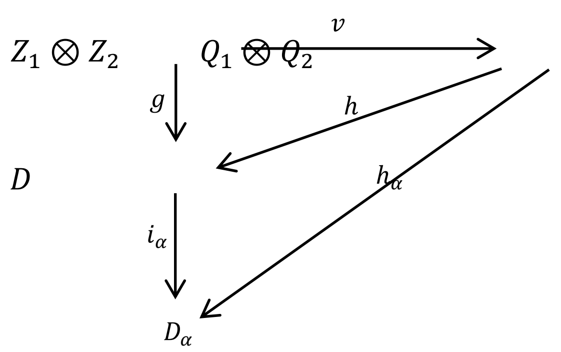
$= g(a)$



Proposition (2.6):
Let $\{D_\alpha\}_{\alpha \in \Delta}$ be family of topological module,
$D = \bigotimes_{\alpha \in \Delta}$ is injective if and only if for all D_α is injective

Proof:

Suppose that D_α is injective, the diagram in figuer 1



(Figuer1)

Such that $z_1 \otimes z_2, Q_1 \otimes Q_2$ are topological module on the same topological ring, $v: Z_1 \otimes Z_2 \rightarrow Q_1 \otimes Q_2$ is injective map,

$g: Z_1 \otimes Z_2 \rightarrow D$ is homomorphism of topological module since D_α is injective topological module then there exists homomorphism of topological module

$h_\alpha: Q_1 \otimes Q_2 \rightarrow D_\alpha$ such that $(h_\alpha \circ v) = i_\alpha \circ g$ .Now define h: $Q_1 \otimes Q_2 \rightarrow D$ define by h$(a_1 \otimes a_2) = =(i_\alpha \circ h)(a_1 \otimes a_2) = \{(i_\alpha \circ h_\alpha)(a_1 \otimes a_2)\}$.If $(a_1 \otimes a_2) = 0 \otimes 0$ It mean that h$(0 \otimes 0)$=

$$= \{i_\alpha \circ h_\alpha(0 \otimes 0)\}$$

$= (0 \otimes 0)$

Hence h is homomorphism topological module.Now if $(a_1 \otimes a_2) \epsilon Z_1 \otimes Z_2$ then

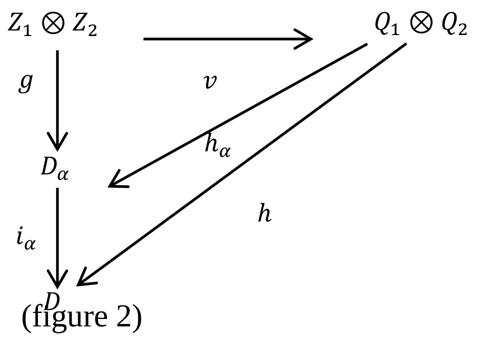
(h $\circ\, v$) $(a_1 \otimes a_2) = \{i_\alpha \circ h_\alpha \circ v(a_1 \otimes a_2)\}$

=$\{i_\alpha \circ p$ $_{\alpha \circ g(a_1 \otimes a_2)\}}$

= $g(a_1 \otimes a_2)$

Conversely:

Suppose that $D = \otimes_{\alpha \in \Delta} D_\alpha$ is injective topological module, the diagram figure 2



(figure 2)

Since D is injective topological module then there exists a homomorphism of topological module $h: Q_1 \otimes Q_2 \rightarrow D$

$h \circ v = i_\alpha \circ g.$

Now define $h_\alpha: Q_1 \otimes Q_2 \rightarrow D_\alpha$

$h_\alpha(b_1 \otimes b_2) = \{(i^{-1} \circ h)(b_1 \otimes b_2)\}$

$(h_\alpha \circ v)(b_1 \otimes b_2) = \{(i^{-1} \circ h \circ v)(b_1 \otimes b_2)\}$

$(h_\alpha \circ v)(b_1 \otimes b_2) = \{(i^{-1} \circ i_\alpha \circ g)(b_1 \otimes b_2)\}$

$= g(b_1 \otimes b_2)$

Hence D_α is injective.

Corollary (2.7):

Let $\{D_\alpha\}_{\alpha \in \Delta}$ be a family of discrat of topological module if $\otimes_{\alpha \in \Delta} D_\alpha$ is injective then any D_α is injective for all $\alpha \in \Delta$.

Proof:

Since D_α is tensor product then by the above Proposition D_α is injective.

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Proposition (2.8):

Let $D_1 \otimes D_2$ be a topological module of a topological ring R and let F be essential extending then inclusion map. $i: D_1 \otimes D_2 \to S_1 \otimes S_2$ extending for injective map from $v: F \to S_1 \otimes S_2$.

Proof:

By the diagram in figure 3

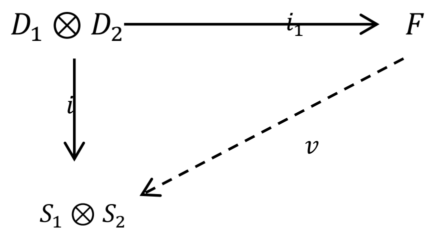


Figure 3

Since $S_1 \otimes S_2$ is injective then there exists homomorphism topological module $v: F \to S_1 \otimes S_2$ such that $v \circ i_1 = i$ and this mean that i extendent to F and
$(D_1 \otimes D_2) \cap kerv = 0$ but F is essential extension for $D_1 \otimes D_2$, $kerv = 0$ that mean $v: F \to S_1 \otimes S_2$ on to homomorphism of topological module.

Since $D_1 \otimes D_2$ open of F and $S_1 \otimes S_2$ then $v: F \to S_1 \otimes S_2$ is injective mapping.

Proposition (2. 9):

Let $D_1 \otimes D_2$ be a left topological module then the following statements are requirement:

1) $D_1 \otimes D_2$ is an injective.
2) If $D_1 \otimes D_2$ open left topological module of F then $F = \otimes (D_1 \otimes D_2)$ and discrete of topological module.
3) If F is essential extension of $D_1 \otimes D_2$ then $D_1 \otimes D_2$has no prove essential extension.

Proof:

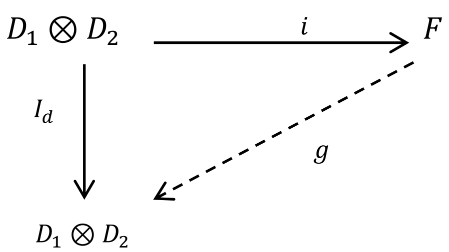
1 ⟶ 2

Suppose that $D_1 \otimes D_2$ open of topological module of, since $D_1 \otimes D_2$ is injective and

$I_d: D_1 \otimes D_2 \to D_1 \otimes D_2$ be a homomorphism topological module by definition of(2.1)

I_d extendengl to homomorphism topological module.

$g: F \to D_1 \otimes D_2$ and the diagram in figure 4 is commutative:



(Figure 4)

Now let $x \in F$ then $g(x) \in D_1 \otimes D_2$

$I_d\big(g(x)\big) = (g \circ i)(g(x))$

It mean that $g(x) = g(g(x)$, but g is homomorphism topological module:

$$g(x - g(x) = 0$$

$$x - g(x) \in kerg$$

$$x \in (D1 \otimes D2) + kerg$$

$$F = (D_1 \otimes D_2) + kerg \Rightarrow (D_1 \otimes D_2) \cap kerg = 0$$

And $F = (D_1 \otimes D_2) \otimes \ kerg$

$kerg$ is discrete

2⇒3

Suppose that $D_1 \otimes D_2 \subset F$ and F is essential extension of $D_1 \otimes D_2$ by the definition of essential extention.

$D_1 \otimes D_2$ is open submodule of F by (2) F is tenser product of $D_1 \otimes D_2$ and discrete submodule of F mean that $(D_1 \otimes D_2) + X = F$ and $(D_1 \otimes D_2) + X = 0$, but F is essential extension for $D_1 \otimes D_2$ and that mean $X = 0$ then $D_1 \otimes D_2 = F$

1⇒3

Suppose that $D_1 \otimes D_2$ has no proper essential extension and let $\acute{F}$ is injective topological module contain open topological submodule $D_1 \otimes D_2$ and let $H_1 \otimes H_2$ be another submodule of F is maximal with the properties

$(H_1 \otimes H_2) \cap (D_1 \otimes D_2) = 0$ it's clear that

$(H_1 \otimes H_2) + (D_1 \otimes D2)/H_1 \otimes H_2 \subset \acute{F}/H_1 \otimes H_2$ by

$(H_1 \otimes H_2) + D_1 \otimes D_2/H_1 \otimes H_2$ requirement topological module with $D_1 \otimes D_2$

We cleam that $\acute{F}/H_1 \otimes H_2$ is essential extension of $(H_1 \otimes H_2) + D_1 \otimes D_2/(H_1 \otimes H_2)$and let $(K_1 \otimes K_2)$ is a submodule of F' contain $D_1 \otimes D_2$ then :

$(K_1 \otimes K_2)/H_1 \otimes H_2$ is a submodule of $\acute{F}/H_1 \otimes H_2$ and suppose that

$\frac{(K_1 \otimes K_2)}{H_1} \otimes H_2 \cap (H_1 \otimes H_2) + \frac{[D_1 \otimes D]}{H_1} \otimes H_2 = 0$ we get :

$(K_1 \otimes K_2) \cap D_1 \otimes D_2 + (H_1 \otimes H_2) \subset D_1 \otimes D_2$ and

$(K_1 \otimes K_2) \cap (D_1 \otimes D_2) = 0$ but $H_1 \otimes H_2$ is a maximal hence

$H_1 \otimes H_2 = K_1 \otimes K_2$ and therefore $(K_1 \otimes K_2)/H_1 \otimes H_2 = 0$

And then $\acute{F}/H_1 \otimes H_2$ is essential extension for

$D_1 \otimes D_2 \cong (H_1 \otimes H_2) + (D_1 \otimes D_2)/H_1 \otimes H_2$ and from the hypothesis $D_1 \otimes D_2$ has no proper essential extension and so:

$(H_1 \otimes H_2) + D_1 \otimes D_2/H_1 \otimes H_2$

$\acute{F}/H_1 \otimes H_2 \cong D_1 \otimes D_2$

$\acute{F} = H_1 \otimes H_2 + D_1 \otimes D_2$

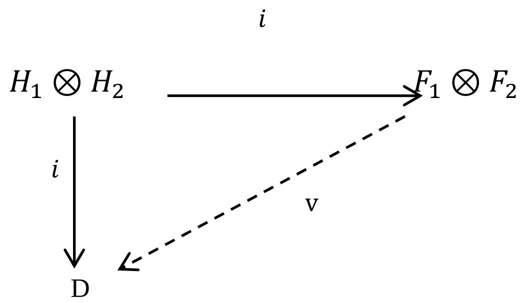
$D_1 \otimes D_2$ is injective

Proposition (2.10):

Let D be injective topological module and let $H_1 \otimes H_2$ be open topological submodule of D not contain in any topological submodule be essential extension of $H_1 \otimes H_2$ then $H_1 \otimes H_2$ is injective.

Proof:

Suppose that $F_1 \otimes F_2$ is essential extension for $H_1 \otimes H_2$, since D is injective from proposition (2.8) there exists injective map: $v: F_1 \otimes F_2 \rightarrow D$, but $f(F_1 \otimes F_2)$ is a submodule of D not contain $H_1 \otimes H_2$ and is essential extension for $H_1 \otimes H_2$ and therefore:
$f(F_1 \otimes F_2) = H_1 \otimes H_2$ and since n is onto then $F_1 \otimes F_2 = H_1 \otimes H_2$.



(Figure 5)

Proposition (2.11):

Let $D_1 \otimes D_2$ be a topological submodule of $S_1 \otimes S_2$ is essential injective for $D_1 \otimes D_2$ then $S_1 \otimes S_2$ has open topological submodule is a maximal essential extension for $D_1 \otimes D_2$.

Proof:

Let Ω be a family of all essential extension for $D_1 \otimes D_2$ and its open topological submodule of $S_1 \otimes S_2$ hence $\Omega \neq 0$ because $D \in \Omega$ by Zorons lemma Ω has maximal element $F_1 \otimes F_2$ we claim that $F_1 \otimes F_2$ is maximal essential extension for $D_1 \otimes D_2$.

Suppose that $\acute{F}_1 \otimes \acute{F}_2$ is a maximal essential extension for $D_1 \otimes D_2$, since $F_1 \otimes F_2$ open of $S_1 \otimes S_2$ and $S_1 \otimes S_2$ is injective by proposition (2.8) there exists injective map:

$v: \acute{F}_1 \otimes \acute{F}_2$ and then $v(\acute{F}_1 \otimes \acute{F}_2)$ element of $S_1 \otimes S_2$ contain $F_1 \otimes F_2$ and is essential extension for $F_1 \otimes F_2$ hence is essential extension for $D_1 \otimes D_2$ and then

$v(\acute{F}_1 \otimes \acute{F}_2) \in \Omega$ but $F_1 \otimes F_2$ is maximal and then $v\left(\acute{F}_1 \otimes \acute{F}_2\right) = F_1 \otimes F_2$ but

$\ker(v) = 0$ and then $\acute{F}_1 \otimes \acute{F}_2 = F_1 \otimes F_2$.

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References

[1] D. Dikranjan, “Recent advances in minimal topography groups,” Topol. Appl., Vol. 85, No. 1–3, pp. 53–91, 1998.

[2] Salih M. A, Majeed T. H, and Salih, M. N., On reasults of quotient for topological modules,periodicals of Engineering and Natural sciences,Vol. 9, No.3, 2021.

[3] Salih M. A, Majeed T. H, and salih M. N., "Tensor Produt on Topologial submodules", Advances in Mechanics,Vol. 9, Issue 3, 2021.

[4] M. M. Choban and R. N. Dumbraveanu, “Functional equivalence of topography spaces and topography modules,” Hacettepe J. Math. Stat., Vol. 46, No. 1, pp. 77–90, 2017.

[5] Majeed, T. H., "On injective Topological Modules", Journal of the College of Basic Education,Vol. 20, No. 20, July 2014.

[6] Majeed, T. H., "On Tenser Product and Direct Sum of Topological Projective Module of Topological Ring"International Journal of Advancedscientific and Technical Resarch,Issue 6, Vol.3, May -June 2016.

[7] Majee, T. H., "Essential Extention of Topological Groups"Magistra No.91, TH.XXVII Maret 2015.