L-open sets in bitopological spaces

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المستخلص:

دراسة نوع جديد من المجموعات المفتوحة في فضاءات التبولوجيا مع دراسة بعض خواصها.

Abstract:

The authors introduce some new open sets in bitopological spaces and study some of their basic properties.

1.Introduction:

The concept of bitopological spaces was initiated by Kelly[1].Let (X,T_1,T_2) be a bitopological spaces. For $A \subset X$ set by $cl_i(A)$ and $\operatorname{int}_i(A)$ we denote the T_i-closure and T_i-interior of A for $i=1,2$.

2.1Definition:

A subset A in bitopological space (X,T_1,T_2) will be termed L -open iff there exists an T_1-open set U such that $U \subset A \subset cl_2(U)$.

The family of all L -open sets in bitopological space (X,T_1,T_2) is denoted by $L.O.(X)$. It's clear that every T_1-open set is L -open but the converse is not true. The complement of L -open set will be called a $L-$closed set.

The following theorems give some properties of L -open sets.

2.2Theorem:

Let (X,T_1,T_2)**be a bitopological space. Let** $A\subset X$ **,** A **is** L **-open iff** $A\subset cl_2(Int_1(A))$**.**

Proof:

Let $A\subset cl_2(Int_1(A))$**.Put** $U=Int_1(A)$**, we have** $U\subset A\subset cl_2(U)$**.**

Conversely. Let A **be** L **-open set. Then** $U\subset A\subset cl_2(U)$ **for some** T_1**-open set** U **.But** $U\subset Int_1(A)$**and thus** $cl_2(U)\subset cl_2(Int_1(A))$**.Hence** $A\subset cl_2(U)\subset cl_2(Int_1(A))$**.**

2.3 Theorem :

Let (X,T_1,T_2) **be a bitopological space. Let** A **and** B **be two subsets of** X **.then**

1) Let $\{A_\alpha\}_{\alpha\in\Lambda}$ **be a collection of** L **-open sets in a bitopological space** X **.then** $\bigcup_{\alpha\in\Lambda}A_\alpha$ **is** L **-open.**

2) Let A **be** L **-open in the bitopological space** X **and suppose** $A\subset B\subset cl_2(A)$**.then** B **is** L **-open.**

Proof:

1) For each $\alpha\in\Lambda$ **,we have an** T_1**-open** U_α **an such that** $U_\alpha\subset A_\alpha\subset cl_2(U_\alpha)$ **.**

Then $\bigcup_{\alpha\in\Lambda}U_\alpha\subset\bigcup_{\alpha\in\Lambda}A_\alpha\subset\bigcup_{\alpha\in\Lambda}cl_2(U_\alpha)\subset cl_2\left(\bigcup_{\alpha\in\Lambda}U_\alpha\right)$**.Hence, let** $U=\bigcup_{\alpha\in\Lambda}U_\alpha$ **. Then** $U\subset\bigcup_{\alpha\in\Lambda}A_\alpha\subset cl_2(U)$ **.**

2) Since A **is** L **-open, there exists an** T_1**-open set** U **such that** $U\subset A\subset cl_2(U)$ **. Then** $U\subset B$**. But** $cl_2(A)\subset cl_2(U)$**and thus** $B\subset cl_2(U)$**. Hence** $U\subset B\subset cl_2(U)$ **and** B **is**

L **-open.**

2.4 Theorem :

Let $\mu=\{B_\alpha\}$ **be a collection of sets in** X **such that,**

1) $T_1 \subset T_2$

2) If $B \in \mu$ **and** $B \subset U \subset cl_2(B)$**, then** $U \in \mu$**. Then** $L.O.(X) \subset \mu$**. Thus** $L.O.(X)$ **is the smallest class of sets in** X **satisfying (1) and (2).**

Proof:

Let $A \in L.0.(X)$**.then** $O \subset A \subset cl_2(O)$ **for some** T_1**-open set** O**. Then** $O \in \mu$ **by (1) and thus** $A \in \mu$ **by (2).**

2.5 Theorem :

Let $A \subset Y \subset X$ **where** X **is a bitopological space and** Y **is a subspace of bitopological space** (X, T_1, T_2)**. Let** $A \in L.O.(X)$**, then.** $A \in L.O.(Y)$

Proof:

$O \subset A \subset cl_2(O)$**, where** O **is** T_1**-open in** X **.now** $O \subset Y$ **and thus** $O = O \cap Y \subset A \cap Y \subset Y \cap cl_2(O)$ **or** $O \subset A \subset cl_{2Y}(O)$ **. Since** $O = O \cap Y$ **,** O **is** $T_1/Y -$**open in** Y **and the theorem is proved.**

2.6 definition :

A Set $M_X \subset X$ **is said to be** L**-neighborhood of a point** $x \in X$ **iff there exists a** $A \in L.O.(X)$ **such that** $A \subset M_X$**.**

2.7 Theorem:

$A \in L.O.(X)$ **Iff** A **is** L**-neighborhood of each** $x \in A$**.**

Proof: easy.

2.8 definition :

A point $x \in X$ **is said to be** L**-limit point of** A **iff for each** $U_x \in L.O.(X)$**,** $U_x \cap A/\{x\} \neq \phi$**.**

The following theorems give some properties of L**-limit points.**

2.9 Theorem :

A **is** L**–closed iff it contains the set of its** L**-limit points.**

Proof: Easy

The set of all L-limit points of A is said to be the L-derived set of A and is denoted by $L-der(A)$.

2.10 Theorem :

If A, B by subsets of bitopological space (X,T_1,T_2). Then,

(1) if $A \subset B$,then $L-der(A) \subset L-der(B)$.

(2) $L-der(A) \bigcup L-der(B) \subset L-der(A \bigcup B)$.

(3) $L-der(A \bigcap B) \subset L-der(A) \bigcap L-der(B)$.

(4) $L-der(L-der(A))/A \subset L-der(A)$.

(5) $L-der(A \bigcup L-der(A)) \subset A \bigcup L-der(A)$.

Proof:

We prove parts (4),(5), and the others follow directly from definitions.

(4) Let $P \in L-der(L-der(A))/A$ and $U \in L.O.(X)$. Then $U \bigcap L-der(A)/\{P\} \neq \phi$.

Let $q \in U \bigcap L-der(A)/\{p\}$. Now since $q \in L-der(A)$ and $q \in U$, $U \bigcap A/\{q\} \neq \phi$.

Let $r \in U \bigcap A/\{q\}$. Then $r \neq p$ for $r \in A$ and $p \notin A$. Therefore $U \bigcap A/\{p\} \neq \phi$ implies that $p \in L-der(A)$.

(5) Let $b \in L-der(A \bigcup L-der(A))$.If $b \in A$, the result is obvious. So let $b \in L-der(A \bigcup L-der(A))/A$.Then, if $U \in L.O.(X)$ containing b, $U \bigcap L-der(A \bigcup L-der(A))/\{b\} \neq \phi$, then $U \bigcap L-der(A)/\{b\} \neq \phi$. Now it follows similarly from (4) that $U \bigcap A/\{b\} \neq \phi$. Therefore $b \in L-der(A)$.

Thus in any case $L-der(A \bigcup L-der(A)) \subset A \bigcup L-der(A)$.

2.11 definition :

Let A be a subset of a bitopological space (X,T_1,T_2) , $A \bigcup L-der(A)$ is defined to be the L-closure of A and is denoted by $L-cl(A)$.

The following theorem gives some properties of L-closure sets.

2.12 Theorem :

Let (X, T_1, T_2) **be a bitopological space. Let** A **and** B **be two subsets of** X **. Then**

(1) $A \subset L-cl(A) = A \bigcup L-der(A)$

(2) If $A \subset B$**, then** $L-cl(A) \subset L-cl(B)$**.**

(3) $L-cl(A) \bigcup L-cl(B) \subset L-cl(A \bigcup B)$**.**

(4) $L-cl(A \bigcap B) \subset L-cl(A) \bigcap L-cl(B)$**.**

(5) $L-cl(\phi) = \phi$ **.** $L-cl(X) = X$ **.**

(6) $L-cl(L-cl(A)) = L-cl(A)$**.**

(7) A **is a** L**–closed iff** $L-cl(A) = A$**,** $L-cl(A)$ **is** L**–closed.**

(8) $L-cl(A) = \bigcap\{F, F \quad is \quad L-closed\ and \quad A \subset F\}$ **, (** $L-cl(A)$ **is the smallest** L**–closed set containing** A **).**

Proof:

We prove parts (2),(6),(7) and the others follow directly from definitions.

(2) Since $A \subset B$ **and by theorem 2.10 part (1)** $L-der(A) \subset L-der(B)$**. Then** $A \bigcup L-der(A) \subset B \bigcup L-der(B)$**, therefore** $L-cl(A) \subset L-cl(B)$**.**

(6)

$$L-cl(L-cl(A)) = L-cl(A \bigcup L-der(A)) = (A \bigcup L-der(A)) \bigcup L-der(A \bigcup L-der(A)) = (A \bigcup L-der(A)) = L-cl(A).$$

(7) By theorem 2.9, A **is** L **-closed iff** $L-der(A) \subset A$ **,i.e., iff** $L-cl(A) = A$ **.**

We introduce the following definition of the L**–interior of a set which is similar to that of standard interior.**

2.13 definition :

A point $x \in X$ **is said to be a** L**–interior point of** A **iff there exists** $U \in L.O.(X)$ **containing** x **, such that** $U \subset A$**. The set of all** L**–interior points of** A **is said to be the** L**–interior of** A **and is denoted by** $L-\text{int}(A)$**.**

The following theorem gives properties of L**–interior sets.**

2.14 Theorem :

For any subsets A, B **of bitopological space** (X, T_1, T_2) **.Then,**
(1) $L-\text{int}(A)$ **is** L**-open.**
(2) $L-\text{int}(A)$ **is the largest** L**-open set contained in** A **.**
(3) A **is** L**-open iff** $A = L-\text{int}(A)$**.**
(4) $L-\text{int}(L-\text{int}(A)) = L-\text{int}(A)$**.**
(5) $L-\text{int}(A) = A/L-der(X/A)$**.**
(6) I) $X/L-\text{int}(A) = L-cl(X/A)$**. II)** $L-\text{int}(X/A) \subset X/L-\text{int}(A)$**.**
(7) I) $X/\ L-cl(A) = L-\text{int}(X/A)$**. II) If** $A \subset B$**, then** $L-\text{int}(A) \subset L-\text{int}(B)$ **.**
(8) I) $L-\text{int}(A) \bigcup L-\text{int}(B) \subset L-\text{int}(A \bigcup B)$ **II)** $L-\text{int}(A \bigcap B) \subset L-\text{int}(A) \bigcap L-\text{int}(B)$**.**
(9) I) $\bigcup_{\alpha \in \Lambda} L-\text{int}(A_\alpha) \subset L-\text{int}\left(\bigcup_{\alpha \in \Lambda} A_\alpha\right)$**.**
II) $L-\text{int}(A/B) \subset L-\text{int}\,(A)/B \subset L-\text{int}\,(A)/L-\text{int}(B)$**.**

Proof:

We prove part (1),(2),(5),(6) and the others follow directly from definitions.

(1) Let $x \in L-\text{int}(A)$**. Then** $U \subset A$ **for some** $U \in L.O.(X)$ **, containing** x **. Also** $y \in U$ **, then** $y \in L-\text{int}(A)$**, therefore** $U \subset L-\text{int}(A)$**. Hence** $L-\text{int}(A)$ **is**

L **–neighborhood of** x**. Therefore by theorem 2.7,** $L-\text{int}(A)$ **is** L**–open.**

(2) Let $V \in L.O.(X)$**,** $V \subset A$ **then** $y \in V$ **,implies that** $y \in A$ **, so that** $y \in L-\text{int}(A)$**.**

Therefore $V \subset L-\text{int}(A)$**. Now the result follows from part (1).**

(5) Let $x \in A$ **. Then** $x \notin L-der(X/A)$**, then exists** L**–open set** U **containing** x **such that** $U \cap X/A = \phi$ **,implies that** $x \in U \subset A$ **,then** $x \in L-\text{int}(A)$**.**

Conversely, let $x \in L-\text{int}(A)$**, then** $x \in L-der(X/A)$ **for** $L-\text{int}(A)$ **is** L**–open and** $L-\text{int}(A) \cap X/A = \phi$**. Therefore** $L-\text{int}(A) = A/L-der(X/A)$**.**

(6) I) $X/L-\text{int}(A) = X/(A/L-der(X/A)) = (X/A) \cup L-der(X/A) = L-cl(X/A)$**.**

II) $L-\text{int}(X/A) \subset X/A \subset L-cl(X/A) = X/L-\text{int}(A)$**.**

3 L–Boundary, L-Exterior and L–Frontier Operators :

In this section we define L**–boundary,** L**-exterior and** L**–frontier of a set ,we study these three operators and prove some standard results.**

The following are elementary definitions which are used throughout the work.

3.1 Definition :

Let A **be a subset of a bitopological space** (X,T_1,T_2)**,** $A/L-\text{int}(A)$ **is said to be the** L**–boundary of** A **and is denoted by** $L-b(A)$ **.**

3.2 Definition :

Let A **be a subset of a bitopological space** (X,T_1,T_2)**,** $L-cl(A)/L-\text{int}(A)$ **is said to be the** L**–frontier of** A **and is denoted by** $L-fr(A)$**.**

3.3 Definition :

Let A **be a subset of a bitopological space** (X,T_1,T_2)**,** $L-\text{int}(X/A)$ **is said to be the** L**-exterior of** A **and is denoted by** $L-ext(A)$**.**

3.4 Theorem :

For any subsets A **,** B **of bitopological space** (X,T_1,T_2)**. Then,**

(1) I) $A = L-\text{int}(A) \cup L-b(A)$**,**

II) $L-\text{int}(A) \cap L-b(A) = \phi$ **,**

(2) A **is** L**–open iff** $L-b(A)=\phi$**,**

(3) $L-b(L-\text{int}(A))=\phi$**,**

(4) $L-\text{int}(L-b(A))=\phi$**,**

(5) $L-b(L-b(A))=L-b(A)$**,**

(6) $L-b(A)=A\cap L-b(X/A)$**,**

(7) I) $L-b(A)=L-der(X/A)$**,**

II) $L-der(A)=L-b(X/A)$**,**

(8) If $A\subset B$ **, then** $L-b(B)\subset L-b(A)$ **,**

(9) I) $L-b(A\cup B)\subset L-b(A)\cup L-b(B)$**,**

II) $L-b(A)\cap L-b(B)\subset L-b(A\cap B)$**,**

Proof:

We prove part (4),(6),(7I),(8) and the others follows directly from definitions and above theorems.

(4)If possible, let $x\in L-\text{int}(L-b(A))$**,then** $x\in L-b(A)$**,also** $L-b(A)\subset A$**.**

Therefore $x\in L-\text{int}(L-b(A))\subset L-\text{int}(A)$**.Hence** $x\in L-\text{int}(A)\cap L-b(A)$**, which contradicts parts (1II). Consequently** $L-\text{int}(L-b(A))=\phi$**.**

(6) $L-b(A)=A/L-\text{int}(A)=A/(X/L-cl(X/A))=A\cap L-cl(X/A)$**.**

(7I) $L-b(A)=A/L-\text{int}(A)=A/(A/L-der(X/A))=L-der(X/A)$**.**

(8) Let $x\in A$**,** $x\in L-b(B)$**, then** $x\notin L-\text{int}(B)$**, implies that** $x\notin L-\text{int}(A)$**.**

Then $x\in L-b(A)$**, therefore** $L-b(B)\subset L-b(A)$**.**

<u>3.5 Theorem :</u>

For any subsets A **,** B **of bitopological space** (X,T_1,T_2)**. Then,**

(1) I) $L-cl(A)=L-\text{int}(A)\cup L-fr(A)$**,**

II) $L-\text{int}(A)\cap L-fr(A)=\phi$**,**

(2) $L-b(A)\subset L-fr(A)$**,**

(3) $L-fr(A)=L-b(A)\cup L-der(A)$**,**

(4) A **is** L**–open iff** $L-fr(A)=L-der(A)$**,**

(5) $L-fr(A)=L-cl(A)\cap L-cl(X/A)$**,**

(6) $L-fr(A)=L-fr(X/A)$**,**

(7) $L-fr(A)$ **is** L**–closed,**

(8) $L-fr(L-fr(A))=L-fr(A)$**,**

Proof:

We prove parts (3),(7),(8) and the others follows directly from definitions and above theorems.

(3) $L-\text{int}(A)\cup L-fr(A)=L-cl(A)=A\cup L-der(A)=L-\text{int}(A)\cup L-b(A)\cup L-der(A)$**.**
Therefore $L-fr(A)=L-b(A)\cup L-der(A)$**.**

(7)
$L-cl(L-fr(A))=L-cl(L-cl(A)\cap L-cl(X/A))\subset L-cl(L-cl(A))\cap L-cl(L-cl(X/A))=$
$L-cl(A)\cap L-cl(X/A)\subset L-fr(A)$.

(8)
$L-fr(L-fr(A))=L-cl(L-fr(A))\cap L-cl(X/L-fr(A))=L-cl(L-fr(A))=L-fr(A)$**.**

<u>3.6 Theorem :</u>

For any subsets A **,** B **of bitopological space** (X,T_1,T_2)**.Then,**

(1) $L-ext(A)$ **is** L**–open set.**

(2) $L-ext(A)=X/\ L-cl(A)$**,**

(3) $L-ext(L-ext(A))=L-\text{int}(L-cl(A))$**,**

(4) I) $L-ext(A\cup B)=L-\text{int}(X/(A\cup B))$**,**

II) $L-ext(A)\cup L-ext(B)\subset L-ext(A\cup B)$**,**

Proof:

We prove parts (2),(3),(4I) and the others follows directly from definitions and above theorems.

(2) $L-ext(A)=L-\text{int}(X/A)=X/L-cl(A)$**.**

(3)
$L-ext(L-ext(A))=L-ext(X/L-cl(A))=X/L-\text{int}(X/L-cl(A))=L-\text{int}(L-cl(A))$**.**

(4I) $L-ext(A\cup B)=L-\text{int}(X/(A\cup B))$
$L-\text{int}((X/A)\cap(X/B))\subset L-\text{int}(X/A)\cap L-\text{int}(X/B)=L-ext(A)\cap L-ext(B)$.

4 Examples :

In this section, we shall show that the converse of the above theorems are not true.

4.1 Remark :

The revers in clusion in the theorem 2.10 par (2),(3) are not true as shown by the following example.

4.2 Example :

Let $X=\{a,b,c\}$ **and** $T_1=\{X,\phi,\{a\},\{b\},\{a,b\}\}$ $T_2=\{X,\phi,\{a\},\{b,c\}\}$**.Then it can be verified that** (X,T_1,T_2) **is bitopological space and** $L.O.(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{b,c\}\}$**.Take**
$A=\{a\},B=\{c\}$**.Then** $L-der(A)=\phi,L-der(B)=\phi$**.Also**
$L-der(A\cup B)=L-der(\{a,c\})=\{c\}\not\subset L-der(A)\cup L-der(B)=\phi$.
Take $A=\{a,c\}$**,** $B=\{b,c\}$**.Then** $L-der(A)=\{c\}$**,** $L-der(B)=\{c\}$**.Again**
$L-der(A\cap B)=L-der(\{c\})=\phi$**.Therefore**
$L-der(A)\cap L-der(B)=\{c\}\not\subset L-der(A\cap B)=\phi$.
Take $A=\{b,c\},B=\{b\}$**.Then** $L-cl(A)=\{b,c\};L-cl(B)=\{b,c\}$**.Therefore**
$L-cl(A)=L-cl(B)=\{b,c\}$**,but** $A\neq B$.
Take $A=\{b\}$ **,** $L-cl(A)=\{b,c\}$**, and** $L-der(L-cl(A))=L-der(\{b,c\})=\{c\}$.
Hence $L-cl(A)=\{b,c\}\not\subset L-der(L-cl(A))=\{c\}$.

4.3 Remark :

The reverse inclusion in theorem 2.12 parts (3) and (4) are not true as shown by the following example.

4.4 Example :

Let $X=\{a,b,c\}$**and** $T_1=\{X,\phi,\{a\},\{b,c\}\}$**,** $T_2=\{X,\phi,\{b\},\{a,c\}\}$**.Then it be verified that** (X,T_1,T_2) **is a bitopological space and** $L.O.(X)=\{X,\phi,\{a\},\{b,c\},\{a,c\}\}$.
Take $A=\{a\},B=\{b\}$**.Then** $L-cl(A)=\{a\},L-cl(B)=\{b\}$**. Again** $A\cup B=\{a,b\}$
and $L-cl(A\cup B)=X$ **. Therefore** $L-cl(A\cup B)=X\not\subset L-cl(A)\cup L-cl(B)=\{a,b\}$.
Take $A=\{b,c\},B=\{a,b\}$**. Then** $L-cl(A)=\{b,c\}$**,** $L-cl(B)=X$ **, and**
$A\cap B=\{b\}$**,** $L-cl(A\cap B)=\{b\}$**. Therefore**
$L-cl(A)\cap L-cl(B)=\{b,c\}\not\subset L-cl(A\cap B)=\{b\}$.

4.5 Remark :

It is obvious that every T_1–open is L-open, but the converse is not true as the following example show.

<u>4.6 Example :</u>

Consider (X,T_1,T_2) defined in example 4.4. Take $A=\{a,c\}$ is L-open set but not
T_1-open .

<u>4.7 Remark :</u>

If $L-\text{int}(A)=L-\text{int}(B)$ does not imply $A=B$ as shown by the following example.

<u>4.8 Example :</u>

Consider (X,T_1,T_2) defined in example 4.2. Take $A=\{a\}, B=\{a,c\}$.
Then $L-\text{int}(A)=\{a\}, L-\text{int}(B)=\{a\}$. Therefore $L-\text{int}(A)=L-\text{int}(B)$ but $A\neq B$.

<u>4.9 Remark :</u>

The revers inclusion in theorem 2.14 parts (8I),(8II) are not true as shown by the following example .

<u>4.10 Example :</u>

Consider (X,T_1,T_2) defined in example 4.2.Take $A=\{c\}, B=\{b\}$.Then

$L-\text{int}(A)=\phi$, $L-\text{int}(B)=\{b\}$,**but** $L-\text{int}(A\cup B)=\{b,c\}$,**therefore**
$L-\text{int}(A\cup B)=\{b,c\}\not\subset L-\text{int}(A)\cup L-\text{int}(B)=\{b\}$.

Let $X=\{a,b,c,d\}$ **and** $T_1=\{X,\phi,\{a\},\{b,c\},\{a,b,c\}\}$,
$T_2=\{X,\phi,\{a\},\{c\},\{a,b\},\{c,d\},\{a,c\},\{a,b,c\},\{a,c,d\}\}$ **. Then it can be verified that** (X,T_1,T_2) **is a bitopological space and**

$L.O.(X)=\{X,\phi,\{a\},\{b,c\},\{a,b,c\},\{a,b\},\{b,c,d\}\}$. **Take** $A=\{b,c\}, B=\{a,b,d\}$ **, then**
$L-\text{int}(A)=\{b,c\}, L-\text{int}\{B\}=\{a,b\}, L-\text{int}(A\cap B)=\phi$ **.Therefore**
$L-\text{int}(A)\cap L-\text{int}(B)=\{b\}\not\subset L-\text{int}(A\cap B)=\phi$.

<u>4.11 Remark :</u>

The revers inclusions in theorem 3.4 parts (9I, II) are not true as shown by the following example.

4.12 Example :

Consider (X, T_1, T_2) **defined in example 4.10. Take** $A = \{a, c\}, B = \{b, d\}$ **, then** $L - b(A) = \{c\}, L - b(B) = \{b, d\}$**.Therefore** $L - b(A) \cup L - b(B) = \{c, b, d\} \not\subset L - b(A \cup B) = \phi$ **.**

Take $A = \{b\}, B = \{a, b, c\}$ **and** $L - b(A) = \{b\}, L - b(B) = \phi, L - b(A \cap B) = \{b\}$**. Then** $L - b(A \cap B) = \{b\} \not\subset L - b(A) \cap L - b(B) = \phi$**.**

4.13 Remark :

If $A \subset B$ **does not imply** $L - fr(A) \subset L - fr(B)$ **as shown by the following example.**

4.14 Example :

Consider (X, T_1, T_2) **defined in example 4.2. Take** $A = \{c\} B = \{b, c\}$ **and** $L - fr(A) = \{c\}, L - fr(B) = \phi$**,therefore** $A \subset B$ **,but** $L - fr(A) = \{c\} \not\subset L - fr(B) = \phi$**.**

4.15 Remark :

The revers inclusion in theorem 3.6 parts (4) are not true as shown by the following example .

4.16 Example :

Consider (X, T_1, T_2) **define in example 4.10. Take** $A = \{a, b\}, B = \{a, c\}$ **, so** $A \cap B = \{a\}$**. Then** $L - ext(A) = \phi, L - ext(B) = \phi . L - ext(A \cap B) = \{b, c, d\}$**.Therefore** $L - ext(A \cap B) = \{b, c, d\} \not\subset L - ext(A) \cup L - ext(B) = \phi$**.**

Reference.

1)Kelly J.C. "Bitopological spaces" Proc.London Math. 13(1963), 71-89.