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The solvability of some semilinear perturbed operator equations in infinite dimensional spaces

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الحلولية لبعض معادلات المؤثر شبه الخطية المقلقلة في الفضاءات غير المنتهية

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المستخلص

في هذا البحث، لقد أُخذ بنظر الاعتبار قابلية الحل ووحدانيته لبعض أصناف معادلات المؤثر شبه الخطية في الفضاءات غير المنتهية. يتكون الجزء الخطي من المعادلات من مؤثر خطي رتيب أعظم مقلقل لتطبيق ثنائي، أما الجزء غير الخطي فهو من نوع ليري شويدر ذو صفة شبه الموجبية أو يحقق بعض الشروط الملائمة في حين تم استخدام فضاء بناخ الانعكاسي الحقيقي وفضاء هلبرت الحقيقي ليكون قاعدة للحلولية.

مفتاح الكلمات: الرتيب الاعظم، التطبيق الثنائي، درجة ليري شويدر، شبه الموجبية، مؤثر لبشز

Abstract

In this paper, the solvability and uniqueness (of the solution) of some classes of semilinear operators equations in infinite dimensional space have been considered. The linearity of the semilinear class is of maximal monotone operator perturbed by duality maping, and the nonlinearity are of Leray-Schauder type operator of quasi-positive or satisfying some suitable conditions. The spaces of solvability are real reflexive Banach space or real Hilbert space.

Keywords: maximal monotone, duality map, Leray-Schauder degree, quasi-positive, Lipschitz operator.

1.Introduction:

In Mortici [4], The semilinear equation of the form $Ax + F(x) = 0$, is considered where A is a linear maximal monotone operator and the nonlinear operator F is of a strongly monotone Lipschitz operator. It is proved that, under these assumptions, the equation $Ax + F(x) = 0$ has a unique solution.

Where in [5],the problem $Ax + F(x) = 0$, where A is a densely defined linear operator, and the nonlinearity F is a quasi-positive operator of Leray-Schauder type have been considered and the existence and uniqueness result under some monotonicity conditions are also obtained as a consequence of the properties of the Leray-Schauder degree. The existence and uniqueness result for the semilinear equation $Au + F(u) = f$, where A is a linear maximal monotone operator and the nonlinearity F is a Lipschitz operator is considered, in [8].

In this paper, it is presented an existence and uniqueness result for some classes of semilinear operator equations in infinite dimensional spaces, where the linearity is a maximal monotone operator perturbed by duality map on a real reflexive Banach space and the nonlinearity is a Leray-Schauder type operator for quasi-positive operator or satisfies some necessary conditions. The solvability approach of this paper are based on Banach fixed point theorem for Leray-Schauder degree theorem as well as Minty-Browder theorem. Some illustrations are also presented with a suitable remarks and discussions .

The following theorem, which is a semilinear perturbed operator equation have been developed.

2.Basics and concepts:

2.1 Definition ,[6]:

Let X be real Banach spaces, and let $A : D(A) \subset X \longrightarrow X^*$ be an linear operator where X^* the dual space to X, then:

(i) A is called monotone if and only if:

$$<Au - Av, u - v> \geq 0$$

(ii) A is called strictly monotone if and only if:

$$<Au - Av, u - v> > 0$$

(iii) A is called strongly monotone if there is a constant c > 0, such that:

$$<Au - Av, u - v> \geq c\|u - v\|^2$$

(iv) A is called coercive if and only if:

$$\lim_{\|u\|\to\infty} \frac{<Au, u>}{\|u\|} \longrightarrow +\infty$$

2.2 Definition, [6]:

Let $A : D(A) \subset H \longrightarrow H$ be an linear operator on the real Hilbert space H, then:

(i) A is called maximal monotone if it is monotone and Au = b if and only if $<b - Av, u - v> \geq 0$, for all $v \in D(A)$ implies A has no proper monotone extension.

(ii) A is called accretive if and only if $(I + \lambda A) : D(A) \longrightarrow H$ is injective and $(I + \lambda A)^{-1}$ is nonexpansive for all $\lambda > 0$.

(iii) A is called maximal accretive if and only if A is accretive and $(I + \lambda A)^{-1}$ exists on H for all $\lambda > 0$.

2.3 Remarks :

Let X, Y be Banach spaces. The following properties for an operator $A : X \longrightarrow Y$ may be define as :

1. A is Lipschitz continuous on $M \subseteq X$, if and only if ther exists a constant $L > 0$ such that:

   $\|Au - Av\| \leq k\|u - v\|$, for all $u, v \in M$ and fixed k.[1]

2. A is k-contractive if and only if A is Lipshitz continuous with and $0 \leq k < 1$.[1]
3. A is nonexpansive if and only if A is Lipschitz continuous with $k = 1$.[10]
4. The operator A is called positive if and only if:

   $\langle Au, u\rangle \geq 0, \forall\, u \in X$. [11]

5- The operator A is called quasi-positive if there exist $\alpha\in$ □ , such that:

$\langle Au,u\rangle \geq \alpha\|Au\|^2$, $\alpha\in$ □ , $u\in X$.[7]

2.4 Lemma ,[11]:

Let $A : D(A) \subset H \longrightarrow H$ be any linear operator on the real Hilbert space H. Then the following three properties of A are mutually equivalent:

(i) A is monotone and $R(I + A) = H$.

(ii) A is maximal accretive.

(iii) A is maximal monotone.

2.5 Theorem (The Riesz theorem),[10]

Let H be a real Hilbert space and let H^* denote the dual space of H , then $f\in H^*$ iff there is a $v\in H$ such that

$$f(v) = \langle v,u \rangle \quad \text{for all } u\in H \qquad (2.1)$$

Here ,the element v of H is uniquely determined by f . In addition

$$\| f \| = \|v\|$$

2.6 Definitions (Duality Map),:

1. L
et H be a real Hilbert space , the duality map J: H $\longrightarrow$ H^* of H through J(v) = f where f is given by (2.1), and
$< f , u > = f(u)$ for all $f \in H^*$ and $u \in H$
Hence $< J(v) , u > = < v , u >$ for all u,v $\in$ H.[10]

2. Set $f(u) = \frac{1}{2} \|u\|^2$, for all u $\in$ X, where X is a real Banach space. The duality map J : X $\longrightarrow$ X* of X is defined to be J = ∂f. [11]

2.7 Remark ,[11]:

1- The duality map J is bijective, continuous and norm preserving, i.e., $\|J(u)\| = \|u\|$, for all u $\in$ X .
2- If X is a real Hilbert space, then J is linear.

2.8 Remarks (1.4.15), [11]:

(i) Let A, B are two monotone operators, such that A : X $\longrightarrow$ X* and B : X $\longrightarrow$ X*, where X is a Banach space. Then A + B : X $\longrightarrow$X* is monotone.

(ii) Let A : D(A) $\subset$ X $\longrightarrow$ X* be maximal monotone, then the mapping A is not empty, i.e., there exists a $(u_0, u_0^*) \in G(A)$, such that $u_0 = 0$ and $u_0^* = 0$, i.e., (0, 0) = G(A).

2.9 Lemma , [11]:

Let b $\in$ X* be given and assume:

(i) C is a nonempty closed convex set in the real reflexive Banach space.

(ii) The mapping A : C $\longrightarrow$ X* is maximal monotone.

(iii) The mapping $B : C \longrightarrow X^*$ is psedomonotone, bounded and dimicontinuous.

Then the original problem $b = Au + Bu$, $u \in C$ has a solution.

2.10 Lemma, [11]:

Let X be a real reflexive Banach space, where X and X* are strictly convex, then the monotone mapping $A : X \longrightarrow X^*$ is maximal monotone if and only if $R(A + J) = X^*$.

2.11 Lemma , [11]:

Let C be a nonempty closed convex subset of the real reflexive B-space X, where X, X* are strictly convex. Suppose that the mapping $A : C \longrightarrow X^*$ is maximal monotone. Then for all $\lambda > 0$, the inverse operator $(A + \lambda J)^{-1} : X^* \longrightarrow X$ is single-values, demicontinuous and maximal monotone.

2.12 Lemma , [11]:

Let X be a real reflexive Banach space, and let the dual space X* be a strictly convex. Set:

$$f(u) = \frac{1}{2}\|u\|, \text{ for all } u \in X$$

Then the duality map $J : X \longrightarrow X^*$ is single-valued, surjective, demicontinuous, maximal monotone, bounded and coercive. For all $u \in X$, we have: $Ju = f'(u)$.

2.13 Lemma ,[5]:

If $F : H \longrightarrow H$ is a quasi-positive operator with $\alpha \in \mathbb{R}$, $\alpha > 1/2$, then

$$\|x - F(x)\| \le \|x\| , \forall\ x \in H,\ x \ne 0.$$

2.14 Lemma , (Compactness of Product), [3]:

Let $A_1 : X \longrightarrow X$ be a compact linear operator and $A_2 : X \longrightarrow X$ be a bounded linear operator (hence continuous). Then A_1A_2 and A_2A_1 are compact.

2.15 Lemma , (Compact Perturbation), [2]:

Let f, g, h : $X \longrightarrow X$ be a mapping of Banach space X, then f is called compact perturbation of the mapping g if and only if f = g + h and h is compact.

2.16 Definition ,[9]:

Let X ,Y be two topological space and let f , g : $X \longrightarrow Y$, we say that f is homotopic to g if there is a map $\hat{H} : [0, 1]\times X \longrightarrow Y$ such that:

$$\left.\begin{aligned}\hat{H}(0,x) &= f\\ \hat{H}(1,x) &= g\end{aligned}\right\} \qquad \forall\, x \in X$$

$[0, 1]\times X=\{(t,x) : t\in[0, 1] , x\in X \}$ and

i.e $\forall\, t\in[0,1]$ let $f_t(x) = \hat{H}(x,t)$ then $\hat{H}$ is called homotopic between f_0 and f_1.

2.17 Lemma ,[5]:

Let f: $U \subseteq H \longrightarrow H$ be such that I – f is compact and let $y \in H \setminus f(\partial U)$.. Then the Leray-Schauder degree d(f,U, y) satisfies the following properties:

(a) If $d(f,U, y) \neq 0$, then $y \in f(U)$.

(b) If $\hat{H} \in C([0,1]\times U,H)$ is such that $I-H(t, \cdot)$ is compact, for all

$t\in[0, 1]$ and $y \in H \setminus \hat{H}([0, 1] \times \partial U)$, then the degree

$d(H(t, \cdot),U, y) = \text{constant}$, $\forall\, t \in [0, 1]$.

(c) The degree for the identity map $I : H \longrightarrow H$ is

$$d(I,U,y) = \begin{cases} 1, & y \in U \\ 0, & y \notin U \end{cases}.$$

3.Solvability of semilinear class by using degree theory

3.1 Theorem :

Let the operator $A: X \longrightarrow X^*$ be a maximal monotone and $J:X \longrightarrow X^*$ a duality map on the real reflexive Banach space X such that X and X* are strictly convex and

$$D(A) \cap \text{int } D(J) \neq 0 \; ; \quad A(0) = 0$$

Then the sum $A + J : X \longrightarrow X^*$ is maximal monotone.

Proof:

On using a translation, we may assume that $0 \in D(A) \cap \text{int } (J)$ by remark (2.8)(ii)

and replacing $u \mapsto Au$ with $u \mapsto Au + c$, for fixed c.

The duality map J is maximal monotone and bounded by Lemma (2.12).

The mapping $A + J : X \longrightarrow X^*$ is monotone, see remark (2.8)(i), and by Lemma (2.10), the mapping A + J is maximal monotone if and only if

$$R(A + 2J) = X^*, \qquad (3.1)$$

i.e., to prove that for all $b^* \in X^*$, the equation

$$b^* = Au + 2Ju, \quad u \in X \qquad (3.2)$$

has a solution.

Replacing $u \mapsto Ju$ with $u \mapsto Ju - b^*$.

It is sufficient to prove the equation:

$$0 = Au + 2Ju, \quad u \in X, \tag{3.3}$$

has a solution.

To obtain a solution u of (3.1) it is sufficient to find a (u, b), such that:

$$\begin{aligned} -b &= (A + 2^{-1}J)u \\ b &= (J + 2^{-1}J)u \end{aligned} \tag{3.4}$$

Where (u, b) $\in$ X×X*.

To this and set:

$$Eb = -(A + 2^{-1}J)^{-1}(-b)$$

$$Fb = (J + 2^{-1}J)^{-1}(b)$$

By the Lemma (2.11), the operator E , F : X* $\longrightarrow$ X are monotone and demicontinuous and we have:

$$R(F) = D(J + 2^{-1}J) = D(J)$$

and hence R(F) is bounded and 0 $\in$ int R(F).

To solve problem (3.2) it is sufficient to solve the equation:

$$Eb + Fb = 0, \ b \in X^*. \tag{3.5}$$

By Lemma (2.9), the equation (3.5) has a solution, implies the desired maximal monotone of A + J.

Hence, the operator E + F : X* $\longrightarrow$ X is monotone and demicontinuous $\Rightarrow$ E + F is maximal monotone .$\Rightarrow$ A+J is maximal monotone .

Based on the result of theorem (3.1) and lemma (2.13), the following theorem is developed and it is needed in the applications .

3.2 Theorem,:

Let A : D(A)$\subseteq$ H $\longrightarrow$ H, linear, maximal monotone operator, and let J : H $\longrightarrow$ H be a duality map on the real Hilbert space H and they are satisfying the conditions of theorem (3.1), where $F : H \longrightarrow H$ be an Leray-Schauder type - operator such that

$$< F(x), x > \geq \alpha\|F(x)\|^2 , \quad \forall\ x \in H \ \ x \neq 0$$

for some $\alpha > 1/2$. Then the equation

$$Ax + J(x) + F(x) = 0 \qquad (3.6)$$

has at least one solution $x \in D(A+J)$.

proof:

By the theorem (3.1),the sum operator (A+J) is a maximal monotone, and by lemma (2.4) it is maximal accretive, hence for all $\lambda > 0$, the operator $(I + \lambda(A+J))$ is invertible with continuous inverse

$(I + \lambda(A+J))^{-1} : H \longrightarrow H$, and by Remarks (2.3) one get

$$\|(I + \lambda(A+J))^{-1}\| \leq 1. \qquad (3.7)$$

Now, the equation (3.6) can be written as

$$(I + A + J)x + (-I + F)x = 0 \Rightarrow$$

$$(I + A + J)x = (I - F)x \Leftrightarrow x = (I + A + J)^{-1}(I - F)x \qquad (3.8)$$

Since F is an operator of Leray-Schauder dgree (see [9]), then $I - F$ is compact and

$$(I + A+J)^{-1}(I - F)$$

is compact, lemma(2.14).

And so on, one can get

$$(I - (I + A+J)^{-1}(I - F))$$ (3.9)

Is compact perturbation of identity map by lemma(2.15) , and hence

$$(I - (I + A+J)^{-1}(I - F)) = 0$$ (3.10)

Can be solved using Leray-Schauder degree as follows:

Since

$$Ax + J(x) + F(x) = 0 \iff (I - (I + A+J)^{-1}(I - F))(x) = 0$$ (3.11)

Let B = B(0, r) be such that $\overline{B} \in D(A+J)$

On using homotopy function we have that

$$\widehat{H}(t, x) = x - (I + A+J)^{-1}(I - F)(x), \; x \in \overline{B}, \; t \in [0, 1].$$ (3.12)

If $0 \in \widehat{H}(1, \partial B)$, the conclusion follows immediately. In order to use the invariance to homotopy of the Leray-Schauder degree (see [9]), we prove that $0 \notin H([0, 1), \partial B)$.

Let us suppose by contrary that $\widehat{H}(t, x) = 0$, for some $x \in \partial B$

and $t \in [0, 1)$.from equation (3.12)

$$\|x\| = t\|(I + A+J)^{-1}(I - F)(x)\|$$

$$\leq \|(I + A+J)^{-1}\| . \| x - F(x)\|$$

By equation (3.7) and lemma (2.13) we get

$$\|x\| \leq \| x - F(x) \| \leq \| x \| .$$

We must have equalities all over, in particular $(I + A+J)^{-1}(I - F)(x) = 0$

Hence $x = 0 \in \partial B$, contradiction. This means that $0 \notin \widehat{H}([0, 1], \partial B)$ and further, (see [9] and lemma (2.17))

$$d(H(1, \cdot), B, 0) = d(H(0, \cdot), B, 0) \Rightarrow$$

$$\Rightarrow d(I - (I + A+J)^{-1}(I - F)(x), B, 0) = d(I, B, 0) = 1.$$

In conclusion, $d(I - (I + A+J)^{-1}(I - F)(x) , B, 0) \neq 0$.

By Leray-Schauder degree, one gets:

The equation $(I -(I + A+J)^{-1}(I - F)(x)) = 0$ and equivalent, the equation

$$Ax + J(x) + F(x) = 0$$

has at least one solution in $D(A+J)$. ■

Illustrations(3.3):

Let $\Omega \in R^n$ be open set and bounded and let $aij \in C^1(\overline{\Omega})$, $1 \leq i, j \leq n$ be real valued functions satisfying the ellipticity property

$$\sum_{|i|,|j|=k} a_{ij}(x)\xi_i\xi_j \geq 0 \qquad \forall\ \xi = (\xi_1,\ldots,\xi_n) \in R^n$$

Let us consider the following elliptic problem

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x i}\left(a_{ij}(t)\frac{\partial x}{\partial x_i}\right) + g(t,x) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(3.19)

The nonlinear part g(t,x) $= \sum_{i=1}^{n} g_i(t,x)$, where $g_i(t, x) = a_{0i}(t)x$, with $a_{0i} \in C(\overline{\Omega})$, $a_0 > p > 0$, under the assumption that the sum of the nonlinear part is quasi – positive [see lemma (3.3)],

First , one have to prove that

$$< \sum_{i=1}^{n} g_i(t,x) , x(t) > \; \geq \alpha^* \sum_{i}^{n} < g_i(t,x), g_i(t,x) >$$

for some $\alpha^* = \min\{\alpha_1, \alpha_{2,\ldots,}\, \alpha_{n}\},\ \alpha^* > 1/2$.

It should be noticed that:

$$\sum_{i=1}^{n} g_i(t,x) = g_1(t,x) + g_2(t,x) + \ldots + g_n(t,x)$$

$$= a_{01}(t)x + a_{02}(t)x + \ldots + a_{0n}(t)x$$

$$= (a_{01}(t) + a_{02}(t) + \ldots + a_{0n}(t)).x$$

$$= a_0(t)x$$

Such that $a_0(t) = \sum_{i}^{n} a_{0i}(t)$.

Hence

$$\int_{\Omega}\sum_{i}^{n} g_i(t,x(t))x(t)dt \ge \alpha^* \sum_{i=1}^{n}\int_{\Omega} g_i^2(t,x(t))dt$$

$$\int_{\Omega} a_0(t).x(t)x(t)dt \ge \alpha^* \left(\int_{\Omega} g_1^2(t,x(t))dt + \int_{\Omega} g_2^2(t,x(t))dt + \ldots + \int_{\Omega} g_n^2(t,x(t))dt \right)$$

$$\int_{\Omega} a_0(t).x^2(t)dt \ge \alpha^* \left(\int_{\Omega} \left(g_1^2(t,x(t)) + g_2^2(t,x(t)) + \ldots + g_n^2(t,x(t)) \right) dt \right)$$

$$\int_{\Omega} a_0(t).x^2(t)dt \ge \alpha^* \left(\int_{\Omega} \left(a_{01}^2(t)x^2(t) + a_{02}^2(t)x^2(t) + \ldots + a_{0n}^2(t)x^2(t) \right) dt \right)$$

$$\int_{\Omega} a_0(t).x^2(t)dt \ge \alpha^* \left(\int_{\Omega} \left(\left(a_{01}^2(t) + a_{02}^2(t) + \ldots + a_{0n}^2(t) \right) x^2(t) \right) dt \right)$$

$$\int_{\Omega} a_0(t).x^2(t)dt \ge \alpha^* \left(\int_{\Omega} a_0^2 x^2(t)dt \right)$$

$$\alpha^* \le \frac{\int_{\Omega} a_0(t).x^2(t)dt}{\int_{\Omega} a_0^2(t)x^2(t)dt}$$

Is fulled with $\alpha^* < 1/ \|a_0\|$.

If H $=L^2(\Omega)$, where $L^2(\Omega)$ is standing for the class of then ,

$$A = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(t) \frac{\partial x}{\partial x_j} \right), \qquad D(A) = H^2(\Omega) \cap H_0^1(\Omega), \ J = 0$$

and $(\sum_{i=1}^{n} F_i(x))(t) = \sum_{i=1}^{n} g_i(t, x(t))$. The problem (3.19) can be written in the abstract form

$$Ax + \sum_{i=1}^{n} F_i(x) = 0 , \ x \in D(A) \subset L^2(\Omega).$$

We have that:

$$-\int_{\Omega}\sum_{i,j=1}^{n}\frac{\partial x}{\partial x_j}\left(a_{ij}(t)\frac{\partial x}{\partial x_j}\right)\text{x} =$$

$$= -\int_{\Omega}\sum_{i,j=1}^{n} a_{ij}\frac{\partial x}{\partial x_j}x_i x ds + \int_{\Omega} a_{ij}\frac{\partial x}{\partial x_j}.\frac{\partial x}{\partial x_i}dx$$

$$= -\int_{\partial\Omega} Nx.xdx + \int_{\Omega} a_{ij}\frac{\partial x}{\partial x_j}.\frac{\partial x}{\partial x_i}dx$$

Since Nx = 0 on $\partial\Omega$

$$= \int_{\Omega} a_{ij}\frac{\partial x}{\partial x_j}.\frac{\partial x}{\partial x_i}dx$$

$$= -\int_{\Omega}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial x}{\partial xj}\right)\text{x} = < \text{Ax} , \text{x} >$$

Where < . , . > is defined as < x , x > = $\int_{\Omega} x^2(t)dt$

$$\Rightarrow < \text{Ax} , \text{x} > = \int_{\Omega} a_{ij}\frac{\partial x}{\partial x_j}.\frac{\partial x}{\partial x_i}dx \geq 0 ,$$

Now, on what the conditions, the sum of quasi-positive operators is still quasi-positive operator . the following lemma is answers this questions

Proposition(3.4):

The following inequality hold

$$b\sum_{i=1}^{n}\| F_i(x)\|^2 \ge c\,\|\sum_{i=1}^{n} F_i(x)\|^2$$

(3.20)

if and only if $c \le b$, $< F_i(x) , F_j(x) > > 0$, $\forall\ i \ne j$, $F_i(x) \ne 0$ $i \in I$

proof :

$$b\sum_{i=1}^{n}\| F_i(x)\|^2 \ge c\,\|\sum_{i=1}^{n} F_i(x)\|^2 \Leftrightarrow c \le \frac{b\sum_{i=1}^{n}\| F_i(x)\|^2}{\|\sum_{i=1}^{n} F_i(x)\|^2} \le \frac{b\sum_{i=1}^{n}\| F_i(x)\|^2}{<\sum_{i=1}^{n} F_i(x),\sum_{i=1}^{n} F_i(x)>}$$

$$\le \frac{b\sum_{i=1}^{n}\| F_i(x)\|^2}{\sum_{i=1}^{n}\| F_i(x)\|^2 + 2\sum_{i=1}^{n} < F_i(x), F_j(x) >}$$

Where $< F_i(x) , F_j(x) > > 0$,$\forall\ i \ne j$, $j = 1, 2, \ldots ,n$.

$$\le \frac{b\sum_{i=1}^{n}\| F_i(x)\|^2}{\sum_{i=1}^{n}\| F_i(x)\|^2} \le b$$

And hence one can obtained $b\sum_{i=1}^{n}\| F_i(x)\|^2 \ge c\,\|\sum_{i=1}^{n} F_i(x)\|^2$

■

Lemma(3.5):

Let $F_i : H \longrightarrow H$ be a quasi-positive operators on the real Hilbert space H for each

i, i = 1, 2, …, n; with real number $\alpha_i \in$ □ , then $\sum_{i=1}^{n} F_i(x)$ is quasi-positive

if $< F_i(x) , F_i(x) > > 0$, $i \neq j$ and $F_i(x) \neq 0 \; \forall \; x \in H$ .Where $\alpha^* = \min\{\alpha_1 , \alpha_2 , \ldots \alpha_n ,\}$,

$c \leq n\alpha^*$

Proof:

$$< \sum_{i=1}^{n} F_i(x) , x> = <F_1(x), x> + <F_2(x), x> + \ldots + <F_n(x), x>$$

$$\geq \alpha_1\|F_1(x)\|^2 + \alpha_2\|F_2(x)\|^2 + \ldots + \alpha_n\|F_n(x)\|^2$$

$$\geq \alpha^*\|F_1(x)\|^2 + \alpha^*\|F_2(x)\|^2 + \ldots + \alpha^*\|F_n(x)\|^2$$

$$\geq (\alpha^* + \alpha^* + \ldots + \alpha^*) \sum_{i=1}^{n} \| F_i(x) \|^2$$

$$\geq n\alpha^* \sum_{i=1}^{n} \| F_i(x) \|^2$$

$$\geq c \, \| \sum_{i=1}^{n} F_i(x) \|^2 \quad \text{by proposition (3.7)}$$

Hence $< \sum_{i=1}^{n} F_i(x) , \quad x> \quad \geq \quad c \| \sum_{i=1}^{n} F_i(x) \|^2 .$

(3.21) ■

Lemma(3.6):

If $F_i : H \longrightarrow H$ is a quasi-positive operators for all i, i = 1, 2, …, n; with real numbers α_i , c and ; where $c > \frac{1}{2}$, $\alpha^* = \min \{\alpha_1, \alpha_2, \dots, \alpha_n\}$.

Then:

$$\left\| x - \sum_{i=1}^{n} F(x) \right\| \leq \|x\| \qquad (3.22)$$

Proof:

$$\left\| x - \sum_{i=1}^{n} F_i(x) \right\|^2 = < x - \sum_{i=1}^{n} F_i(x), x - \sum_{i=1}^{n} F_i(x) >$$

$$= \|x\|^2 - 2<x, \sum_{i=1}^{n} F_i(x) > + \| \sum_{i=1}^{n} F_i(x) \|^2$$

$$\leq \|x\|^2 - 2n\alpha^* \sum_{i=1}^{n} \|F(x)\|^2 + \| \sum_{i=1}^{n} F_i(x) \|^2$$

$$\leq \|x\|^2 - 2c \| \sum_{i=1}^{n} F_i(x) \|^2 + \| \sum_{i=1}^{n} F_i(x) \|^2 \text{ by lemma (3.13)}$$

$$\leq \|x\|^2 - (2c-1) \sum_{i=1}^{n} \| F_i(x) \|^2 \leq \|x\|^2.$$

■

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