Dual Coclosed Rickart Modules

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Abstract

The dual concept of coclosed rickart modules is defined in this paper. Consider M as a right module over an arbitrary ring R with identity where $\mathrm{S} = \mathrm{End}_R(M)$ is the endomorphism ring of M. We call a module M is dual coclosed rickart when every $f \in \mathrm{End}_R(M)$, $\mathrm{Im}\, f$ is a coclosed submodule of M. Number of conclusions are gained and some connections between these modules and other related modules are studied.

مقاسات كوكلوسد ريكارت الرديفة

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الخلاصة

ليكن M مقاسا ايمن معرف على حلقة R ذات محايد وليكن $\mathrm{S} = \mathrm{End}_R(M)$. قدمنا في هذا البحث الفهوم الرديف لمقاسات كوكلوسد ريكارت. إذ يقال عن المقاس M بأنه كوكلوسد ريكارت الرديفة اذا لكل $f \in \mathrm{End}_R(M)$ فان $\mathrm{Im}\, f$ يكون مقاسا جزئيا كوكلوسد في M . حصلنا على عدد من النتائج حول هذا الفهوم وعلاقته بانواع اخرى من الموديولات.

Key words : dual coclosed rickart modules, dual relatively coclosed rickart modules, dual rickart modules, cosemisimple modules, modules with CCSP.

1 Introduction

Following Ghaleb [2], M is told coclosed rickart whenever every $f \in \mathrm{End}_R(M)$, $\mathrm{Ker}\, f$ is coclosed of M. A submodule N of M is said small in M whenever $K \leq M,\ N + K = M$ yield $K = M$. We say a submodule L of a module M is coclosed in M when $\frac{L}{K} \ll \frac{M}{K}$ then $L = K$ for each $K \subseteq L$ [1]. Equivalently, for each proper submodule $K \subset L$, there is a submodule N

of M where $L + N = M$ while $K + N \neq M$ iff L is coclosed of M. In [3] a generalization of dual rickart modules is presented by using the concept of purity. Our purpose of the paper is to consider the dual concept of coclosed rickart modules as another generalization of dual rickart modules. We name M is dual coclosed rickart when each $f \in \mathrm{End}_R(M)$, $\mathrm{Im}f$ is a coclosed submodule in M. Other studies in [5],[6],[7],[8],[9],[10],[11] and [12] is related topics.

The paper contains three parts. In part two, we investigate concept of dual coclosed rickart modules and supply basic properties of this concept. We see that direct summands of dual coclosed rickart modules gain the property (Proposition 2.6), this is not so for direct sums (Remark 2.7). We get a condition which allow direct sums of dual coclosed rickart modules to be dual coclosed rickart (Proposition 2.8). We look for any connection between dual coclosed rickart modules and other modules. We see that dual coclosed rickart modules and dual rickart modules coincide in lifiting modules (Proposition 2.12). The concept of relatively dual coclosed rickart modules is presented and studied in section three. By using the CCSP, we will provide a condition for modules to be relatively dual coclosed rickart (for example, Theorem 3.3, Proposition 3.10) where a module M is named to be gain coclosed sum property (in short CCSP) when the sum of two coclosed submodules of M is coclosed [4]. Many results are investigated, we find that family of rings R for which each right R-module is relatively dual rickart is right cosemisimple (Proposition 3.12).

2 Dual Coclosed Rickart Modules

Dual coclosed rickart modules is studied within this part. Basic facts of this type of modules are investigated. We begin with the next.

Definition 2.1. consider M as a module over R. We call M is dual coclosed rickart when each $f \in \mathrm{End}_R(M)$, $\mathrm{Im}f$ is a coclosed submodule of M.

Remarks with Examples 2.2.

(1) Clearly each cosemisimple module is dual coclosed rickart but not conversely, where M is told cosemisimple whenever any submodule of M is coclosed [1]. For example, the module $\mathbb{Q}$ as $\mathbb{Z}$-module is dual coclosed rickart since every endomorphism of $\mathbb{Q}$ is either zero or an isomorphism but its not cosemisimple.

(2) obviously each dual rickart module is dual coclosed rickart but not conversely. Discuss the ring $R = \prod_{i \geq 1} \mathrm{F}_i$ with $\mathrm{F}_i = \mathrm{F}$ is a field for each $i \geq 1$. Obviously R is

not semisimple then by [10, Theorem 2.24], find a module M may not Rickart. beside this, R is commutative regular (in sense von Neumann) then by [1], R is cosemisimple (V-ring) implies that $\text{Rad}\left(\frac{M}{N}\right) = 0$ where N a submodule of M. Hence any submodule is coclosed, therefore M is dual coclosed rickart.

(3) When M is a coclosed simple modul yield M need not be dual coclosed rickart, where a module M is named coclosed simple when $M \neq \{0\}$ and it is gain no coclosed submodules except $\{0\}$ and M [4]. For example, $\mathbb{Z}_4$ as $\mathbb{Z}$-module is coclosed simple, while it is not dual coclosed rickart since there exists an endomorphism $f : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4$ by $f(m) = m2$ for each $m \in \mathbb{Z}_4$, thus $\text{Im} f = \{\bar{0}, \bar{2}\}$ is not a coclosed submodule in $\mathbb{Z}_4$.

(4) When M is coquasi-Dedekind module yield it is dual coclosed Rickart, where modul $M \neq 0$ is called coquasi-Dedekind when every $0 \neq f \in \text{End}_R(M)$, $\text{Im} f = M$ [14]. The reverse is not hold as follows. $\mathbb{Z}_6$ as $\mathbb{Z}$-module is dual coclosed rickart whereas it is not coquasi-Dedekind.

(5) When M is a dual coclosed rickart coclosed simple over R, implies it is coquasi-Dedekind.

Proof. Let $0 \neq M$ be a dual coclosed rickart over R, $0 \neq f \in \text{End}_R(M)$, implies $\text{Im} f$ is a coclosed, but M is coclosed simple implies that $\text{Im} f = M$. This mean M is coquasi-Dedekind.

(6) M is dual coclosed rickart, implies M may not be coclosed rickart. For example, $\mathbb{Z}_{p^\infty}$ over $\mathbb{Z}$ such that p is a prime, it is not difficult find $\text{Im} f = \mathbb{Z}_{p^\infty}$ for each $f \in \text{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$, in fact, $\mathbb{Z}_{p^\infty}$ is coquasi-Dedekind then by Remark 4, it is a dual coclosed rickart. But $\mathbb{Z}_{p^\infty}$ is not coclosed rickart because there exists an endomorphism $f : \mathbb{Z}_{p^\infty} \longrightarrow \mathbb{Z}_{p^\infty}$ defined by $f\left(\frac{n}{p^m} + \mathbb{Z}\right) = \frac{n}{p^{m-1}} + \mathbb{Z}$, for each $n \in \mathbb{Z}$ and m is a positive integer implies $\ker f = < \frac{n}{p} + \mathbb{Z} >$ is not a coclosed submodule in $\mathbb{Z}_{p^\infty}$.

(7) When M is coclosed rickart, implies it is may not be dual coclosed rickart. Such as $\mathbb{Z}$ is coclosed rickart since $\ker f = 0$ for every $f \in \text{End}_{\mathbb{Z}}(\mathbb{Z})$,. But $\mathbb{Z}$ is not dual coclosed rickart since for any endomorphism $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ via $f(m) = nm$ for each $m \in \mathbb{Z}$, $\text{Im} f = n\mathbb{Z}$ not coclosed in $\mathbb{Z}$, with n an integer greater than one.

Proposition 2.3. The dual coclosed rickart property under an isomorphism is translated.

Proof. Let M_1 and M_2be modules over R, M_1 is dual coclosed rickart with $f : M_1 \to M_2$ is an isomorphism. Assume $g \in \mathrm{End}_R(M_2)$, we prove that $\mathrm{Im}g$ is coclosed submodule in M_2. Study K is any proper submodule in M_2 , $K \subset \mathrm{Im}g$, $f^{-1}(K) \subset f^{-1}(\mathrm{Im}g)$. But $f^{-1}(\mathrm{Im}g)$= Im ($f^{-1}gf$), to show this. Let $x \in f^{-1}(\mathrm{Im}g)$, $x = f^{-1}(y)$, $y \in \mathrm{Im}g$ so there is $m \in M_2$ such that $g(m) = y$. On the other hand, because f is an isomorphism, there exist $n \in M_1$ and $f(n) = m$. It follows that $x = f^{-1}gf(n)$, this means that $x \in$ Im ($f^{-1}gf$). The reverse inclusion is clear. This means that $f^{-1}(K) \subset$ Im ($f^{-1}gf$), but $f^{-1}gf \in \mathrm{End}_R(M_1)$ and M_1 is dual coclosed rickart then Im ($f^{-1}gf$) is coclosed submodule and so there exist a submodule N of M_1 , $f^{-1}(\mathrm{Im}g) + N = M_1$ but $f^{-1}(K) + N \neq M_1$. This means that $\mathrm{Im}g + f(N) = M_2$ but $K + f(N) \neq M_2$, that is $\mathrm{Im}g$ is a coclosed submodule in M_2. Hence M_2 is dual coclosed rickart.

Examples 2.4.

(1) M is dual coclosed rickart with N a submodule, found N need not be dual coclosed rickart, as the following. $\mathbb{Q}$ as $\mathbb{Z}$-module is dual coclosed rickart whereas the submodule $\mathbb{Z}$ of $\mathbb{Q}$ is not a dual coclosed rickart as $\mathbb{Z}$-module. Another example, $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module is dual coclosed rickart**,** let $N = < \frac{1}{p^2} + \mathbb{Z} >$ be the submodule of $\mathbb{Z}_{p^\infty}$ generated by $\frac{1}{p^2} + \mathbb{Z}$. Easily to check $N \cong \mathbb{Z}_{p^2}$ not dual coclosed rickart over $\mathbb{Z}$ for each prime number p.

(2) If each proper submodule in M is dual coclosed rickart, yield M need not be dual coclosed rickart. See $\mathbb{Z}_4$ over $\mathbb{Z}$ in which every proper submodule is simple module, so they are dual coclosed rickart, while $\mathbb{Z}_4$ is not dual coclosed rickart.

We record the next from (14).

Lemma 2.5. Assume M is a module over R together with $K \subset N$ submodules in M. when K coclosed of M, yield K coclosed of N. The reverse is hold when N is coclosed of M.

Proposition 2.6. Each summand of a dual coclosed rickart module is dual coclosed rickart.

Proof. Assume M is a dual coclosed rickart over R with A a summand, then $M = A \oplus B$ where B a submodule. Let $f \in \mathrm{End}_R(A)$, then we have the following $A \oplus B \xrightarrow{\rho} A \xrightarrow{f} A \xrightarrow{i} M$. Say $g = i f \rho$, then $g \in \mathrm{End}_R(M)$, implies Im g is coclosed in M.We possess $g(M) = (i f \rho)(M) = (i f)(A) = i(f(A)) = f(A)$, yield Im g = Im f.

This means $\operatorname{Im} f$ is coclosed. But A is containing $\operatorname{Im} f$, therefore by lemma 2.5, $\operatorname{Im} f$ is coclosed of A. Hence A is a dual coclosed rickart module over R.

Remark 2.7. A Direct sum of dual coclosed rickart modules is not necessary dual coclosed rickart. For example, $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ as $\mathbb{Z}$-module. Consider $f : M \longrightarrow M$ via $f\left(\frac{m}{p^t} + \mathbb{Z}, \bar{n}\right) = \left(\frac{n}{p^t} + \mathbb{Z}, \bar{0}\right)$ where $m \in \mathbb{Z}$, $t = 0,1,2,\ldots$ and $\bar{n} \in \mathbb{Z}_p$. Then $\operatorname{Im} f \cong \mathbb{Z}_p \oplus \bar{0}$ which is not coclosed in $\mathbb{Z}_{p^\infty} \oplus \bar{0}$, and hence it is not coclosed in M. Therefore $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ is not dual coclosed rickart, while $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_p$ are dual coclosed rickart.

Recall a submodule N of M is named fully invariant when $f(N)$ is included in N for each $f \in \operatorname{End}_R(M)$ (16). We name M duo when each submodule is fully invariant [13].

Proposition 2.8. Assume $M = \bigoplus_{i \in I} M_i$ is a duo R-module for an index set Λ. M is a dual coclosed rickart iff M_i is a dual coclosed rickart for each $i \in \Lambda$.

Proof. The first side according to Proposition 2.6. For the convers, let $M = \bigoplus_{i \in I} M_i$ and $f = (f_{ij}) \in \operatorname{End}_R(M)$, $f_{ij} \in \operatorname{Hom}_R(M_j, M_i)$. Since M_i is fully invariant in $M = \bigoplus_{i \in I} M_i$, follows $\operatorname{Hom}_R(M_j, M_i) = 0$ for each $i \neq j$ [13, Lemma 1.9]. Further $f(M_i) \subseteq M_i$ for all $i \in \Lambda$, this implies that $\operatorname{Im} f = \bigoplus_{i\in\Lambda} \operatorname{Im} f_{ii}$. Our assertion is that $\operatorname{Im} f$ a coclosed in M. To show this, assume $\subset \bigoplus_{i\in\Lambda} \operatorname{Im} f_{ii}$, K is fully invariant submodule. From [13, result 2.1]. $K = \bigoplus_{i\in\Lambda} (K \cap M_i)$, let $K_i = K \cap M_i$ for each $i \in \Lambda$. Obviously $K_i \subseteq \operatorname{Im} f_{ii}$. Since M_i is a dual coclosed rickart module implies that $\operatorname{Im} f_{ii}$ is a coclosed in M_i for all $i \in \Lambda$. Thus there is a submodule N_i , $\operatorname{Im} f_{ii} + N_i = M_i$ but $K_i + N_i \neq M_i$. This implies that $(\bigoplus_{i\in\Lambda} \operatorname{Im} f_{ii}) + (\sum_{i\in\Lambda} N_i) = \bigoplus_{i\in\Lambda} M_i$, but $(\bigoplus_{i\in\Lambda} K_i) + (\sum_{i\in\Lambda} N_i) \neq \bigoplus_{i\in\Lambda} M_i$. Put $N = \sum_{i\in I} N_i$. So we have $\operatorname{Im} f + N = M$ but $K + N \neq M$ and hence $\operatorname{Im} f$ is coclosed .

Proposition 2.9. The next are balance

(1) $\oplus_\Lambda R$ is adual coclosed rickart over R for any index set Λ.

(2) All projective modules are dual coclosed rickart.

(3) All free R-modules are dual coclosed rickart modules.

Proof. (1) $\Rightarrow$ (2) Study M as projective over R yield a free module F over R is found with an epimorphism $f : F \longrightarrow M$. Because $F \cong \oplus_\Lambda R$ for some index set Λ. We gain $0 \to \ker f$

$\overset{i}{\rightarrow} \oplus_\Lambda R \overset{f}{\rightarrow} M \rightarrow 0$. But M is projective then the sequence splits. Thus $\oplus_\Lambda R \cong \ker f \oplus M$. Because $\oplus_\Lambda R$ is dual coclosed rickart, therefore via Lemma 5, M is dual coclosed rickart .

(2) ⇒ (1) It is clear and (1) ⇔ (3) Similar proof of (2) ⇔ (1).

Let us see the next condition (*) for a module over R :

For any submodule N of M for which $N \cong H$ where H is a summand in M, yield N coclosed.

Proposition 2.10. The condition (*) is satisfied in each dual coclosed rickart module

Proof. Discuss N, H as two submodules in M and H summand in M with $N \cong H$. So we have $f: H \rightarrow N$ is an isomorphism, $M \overset{\rho}{\rightarrow} H \overset{f}{\rightarrow} N$ where ρ is the natural projection map of M onto H. Let $= f\rho$, then $g \in \text{End}_R(M)$ and $\text{Im}g = f\rho(M) = N$. By assumption, M is dual coclosed rickart and hence N coclosed in M.

Corollary 2.11. Assume M is dual coclosed rickart over R with the condition (*) for each submodule N in M, then M is cosemisimple.

. **Proof.** Obvious via result 2.10.

Proposition 2.12. Each lifiting dual coclosed rickart module is dual rickart.

Proof. Assume M is lifiting coclosed rickart over R with $f \in \text{End}_R(M)$. Because M lifting, find a direct summand K, $K \subseteq \text{Im} f$ and $\frac{\text{Im} f}{K} \ll \frac{M}{K}$. Because M is dual coclosed rickart, implies $\text{Im} f$ is a coclosed, therefore $\text{Im} f = K$, as desired.

Recall M is cohopfian if each monomorphism $f \in \text{End}_R(M)$ is an isomorphism [1].

Proposition 2.13. When M is a dual coclosed rickart coclosed simple over R yield it is cohopfian.

Proof. Let $0 \neq f \in \text{End}_R(M)$ is a monomorphism. Because M is dual coclosed rickart, then $\text{Im}f$ is a coclosed submodule. But M is coclosed simple and $f \neq 0$, so $\text{Im}f = M$. That is f is an epimorphism and hence M is cohopfian.

3 Relatively Dual Coclosed Rickart Modules

Relatively dual coclosed rickart modules is discussed in this place. Basic facts of this modules are presented. The type of right cosemisimple (V-rings) rings R is shown to be exactly that for which each right R-module is relatively coclosed rickart. Our focus, in this part is on the question: When do certain R-modules have the relatively dual coclosed rickart property.

Definition 3.1. We name M is relatively dual coclosed rickart module to other N if each $f \in \mathrm{Hom}_R(M, N)$, $\mathrm{Im} f$ is coclosed of M.

Thus, as special case, M is dual coclosed rickart iff M is relatively dual coclosed rickart to M.

Remarks and Examples 3.2.

(1) One can easily see that when N is cosemisimple R-module, implies each module over R is relatively dual coclosed rickart to N.

(2) When M is cosemisimple module over R then M need not be relatively dual coclosed rickart to an R-module N. For example, $\mathbb{Z}_2$ as $\mathbb{Z}$-module is cosemisimple while it is not relatively dual coclosed rickart to $\mathbb{Z}_4$ as $\mathbb{Z}$-module, since there exists the homomorphism $f : \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ via $f(m) = m2$ for each $m \in \mathbb{Z}_2$. Then $\mathrm{Im}\, f = \{\bar{0}, \bar{2}\}$ which is not coclosed in $\mathbb{Z}_4$.

(3) M is relatively dual coclosed rickart to N, leads N may not be relatively dual coclosed rickart to M. For example, let $\mathbb{Z}_4$ and $\mathbb{Z}$ as $\mathbb{Z}$-modules. Then $\mathbb{Z}_4$ is relatively dual coclosed rickart to $\mathbb{Z}$ for each positive integer n greater than one, in fact $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_4, \mathbb{Z}) = 0$. Also, $\mathbb{Z}$ is not relatively dual coclosed rickart to $\mathbb{Z}_4$, since there exists a homomorphism $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_4)$ defined by $f(m) = m2$ for each $m \in \mathbb{Z}$, implies that $\mathrm{Im} f = \{\bar{0}, \bar{2}\}$ is not coclosed in $\mathbb{Z}_4$.

(4) When M is a dual coclosed rickart module over R, implies M need not be relatively dual coclosed rickart to N as R-module as follows, $\mathbb{Z}_p$ as the $\mathbb{Z}$-module is dual coclosed rickart where p is prime. But $\mathbb{Z}_p$ is not relatively coclosed rickart to $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module because there exists the inclusion homomorphism $i \in \mathrm{Hom}_R(\mathbb{Z}_p, \mathbb{Z}_{p^\infty})$, implies that $\mathrm{Im} i = \mathbb{Z}_p$ which is not a coclosed submodule in $\mathbb{Z}_{p^\infty}$.

(5) If M is relatively dual coclosed rickart to an R-module N, then M may not be dual coclosed rickart. For example, consider the $\mathbb{Z}$-module $\mathbb{Z}_4$ is relatively dual coclosed rickart to the $\mathbb{Z}$-module $\mathbb{Z}_3$, because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_4, \mathbb{Z}_3) = 0$. But $\mathbb{Z}_4$ is not dual coclosed rickart.

(6) When M is a coclosed simple or coquasi-Dedekind module over R, implies M need not be relatively dual coclosed rickart to N as R-module as follows, $\mathbb{Z}_2$ as $\mathbb{Z}$-module is coclosed simple and coquasi-Dedekind but not relatively dual coclosed rickart to $\mathbb{Z}_4$ as $\mathbb{Z}$-module.

Theorem 3.3. The next statements are equivalent

(1) M is relatively dual coclosed rickart to N.

(2) For each submodule B of N, each summand A in M is relatively dual coclosed rickart to B.

(3) For ech summand A of M, for each coclosed submodule B in N and any $f \in \mathrm{Hom}_R(M, B)$, the image of the restricted map $f|_A$ is coclosed of A.

Proof. (1) $\Rightarrow$ (2) Assume M is relatively dual coclosed rickart to N. Assume A is a smmand in M, B a submodule of N. Let $f \in \mathrm{Hom}_R(A, B)$. Study the next $M = A \bigoplus H \overset{\rho}{\rightarrow} A \overset{f}{\rightarrow} B \overset{i}{\rightarrow} N$ for wher H a submodule of M. Say $g = i\, f\, \rho \in \mathrm{Hom}_R(M, N)$. This implies that $\mathrm{Im} g$ is a coclosed submodule in N. Then $g(M) = (i\, f\, \rho)(M) = (i\, f\,)(A) = i\big(f\,(A)\big) = f\,(A)$ and hence $\mathrm{Im} f$ is coclosed in N. But B is containing $\mathrm{Im} f$, thus by lemma 2.5, $\mathrm{Im} f$ is coclosed in B. Therefore A is relatively dual coclosed rickart to B.

(2) $\Rightarrow$ (3) Assume A is a summand of M with B coclosed in N. Let $f \in \mathrm{Hom}_R(M, B)$, implies $f|_A \in \mathrm{Hom}_R(A, B)$. Since A is relatively dual coclosed rickart to B implies $\mathrm{Im}\, f|_A$ is closed in B.

(3) $\Rightarrow$(1) Obviously by taking $A = M$ and $B = N$.

The next two lemmas are in [4].

Lemma 3.4. Discuss M as a module over R. CCSP is gained via M iff each coclosed of M has the CCSP.

Lemma 3.5. Assume M is a module over R gains CCSP, implies for each decomposition $M = A \oplus B$, for each $f \in \mathrm{Hom}_R(A, B)$, $\mathrm{Im} f$ is a coclosed submodule in M.

Theorem 3.6. Assume M is a module over R with CCSP. When $A \oplus B$ is coclosed in M. Yield A is relatively dual coclosed rickart module to B.

Proof. Assume that M has the CCSP. Then by lemma 3.4, every coclosed submodule of M has the CCSP implies that $A \oplus B$ has the CCSP. By lemma 3.5, for every $f \in \operatorname{Hom}_R(A, B)$, $\operatorname{Im} f$ is a coclosed submodule in $A \oplus B$. But $\operatorname{Im} f \subseteq B$ and $\operatorname{Im} f$ is a coclosed submodule in B. Hence A is relatively dual coclosed rickart to B.

As an immediate consequences we have

Corollary 3.7. Assume M and N as modules over R. If $M \oplus N$ has the CCSP, then M is relatively dual coclosed rickart to N.

Corollary 3.8. When $M \oplus M$ gains CCSP, implies M is dual coclosed rickart module.

Remark 3.9. The innverse of Result 3.7 is not hold any time. Discuss $\mathbb{Z}_4$ as $\mathbb{Z}$-module is relatively dual coclosed rickart to $\mathbb{Z}_2$ over $\mathbb{Z}$. While $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ over $\mathbb{Z}$ does not have the CCSP. To show this, let $A = (\bar{1}, \bar{0})\mathbb{Z}$ and $B = (\bar{1}, \bar{2})\mathbb{Z}$ be the submodules generated by $(1, \bar{0})$ and $(\bar{1}, \bar{2})$ respectively. It is clear that A, B are summands in M, implies that A, B are coclosed submodules in M. It is not hard to see that $A + B = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{2}), (\bar{1}, \bar{0}), (\bar{1}, \bar{2})\}$ is not summand of M. Further $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is lifting , implies that $A + B$ is not coclosed in $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

Proposition 3.10. Assume $\{M_i\}_{i \in \Lambda}$ is a set of modules over R where $\Lambda = \{1, 2, \dots, n\}$ and N a module over . Study the next equivalence

(1) If N has the CCSP, then $\bigoplus_{i=1}^{n} M_i$ is relatively dual coclosed rickart to N.
(2) M_i is relatively dual coclosed rickart to N for each $i \in \Lambda$

Proof. (1) $\Rightarrow$ (2) is immediately from Theorem 3.3.

(2) $\Rightarrow$ (1) M_i is relatively dual coclosed rickart to N for all $i = 1, 2, \dots, n$ and N has the CCSP. To show that $\bigoplus_{i=1}^{n} M_i$ is relatively dual coclosed rickart to N, let $f \in \operatorname{Hom}_R(\bigoplus_{i=1}^{n} M_i,\ N)$, $f = (f_i)_{i \in \Lambda}$ and $f|_{M_i} = f_i : M_i \to N$ is an R-homomorphism for each $i = 1, 2, \dots, n$. Thus $\operatorname{Im} f = \sum_{i=1}^{n} f_i(M_i)$. But $\operatorname{Im} f_i$ is a coclosed submodule in N and N has the CCSP, therefore $\operatorname{Im} f = \sum_{i=1}^{n} f_i(M_i)$ is coclosed in N, and hence $\bigoplus_{i=1}^{n} M_i$ is relatively dual coclosed rickart to N.

Corollary 3.11. Assume $\{M_i\}_{i \in \Lambda}$ is a set of modules over R where $\Lambda = \{1, 2, \dots, n\}$. Then the following are equivalent

(1) If M_i has the CCSP for all $j \in \Lambda$, then M_i is relatively dual coclosed rickart to $\oplus_{i=1}^{n} M_i$.

(2) M_i is relatively dual coclosed rickart to M_i for all $i = 1,2, \dots, n$.

Cosemisimple rings via relatively dual coclosed rickart modules over R is characterized via the next.

Proposition 3.12. Discuss next equivalence.

(1) R is cosemisimple right R-module.

(2) All R-modules are cosemisimple.

(3) All R-modules are relatively dual coclosed rickart.

(4) All R-modules have CCSP.

(5) All flat modules over R have CCSP.

(6) Every projective modules over R have CCSP.

Proof. $(1) \Leftrightarrow (2)$ follows by [1, Theorem 1.12], $(2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$and $(2) \Rightarrow (3)$ is obvious.

$(3) \Rightarrow (1)$ Let I be an ideal of R. By assumption, I is relatively dual coclosed rickart to R over R. Then for every $f \in \mathrm{Hom}_R\,(I, R)$, $\mathrm{Im}\, f$ is coclosed in R. Since $i \in \mathrm{Hom}_R\,(I, R)$, implies $\mathrm{Im}\, i = I$ is coclosed in R. Hence R is cosemisimple.

$(6) \Rightarrow (1)$ Consider I as a submodule of R, find a projective module F over R, an epimorphism $\alpha : F \to I$. Let $i : I \to R$ be the inclusion fuction. Then we have $i\alpha : F \to R$. Since $F \oplus R$ is projective. By assumption it has CCSP. By Lemman **5**, $\mathrm{Im}\, i\alpha = \mathrm{Im}\, i = I$ is coclosed of R as asserted.

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