On (K-N)*-quasinormal operator

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Abstract

In this paper, we introduce another type of quasinormal operator is called (K-N)*-quasinormal operator and give some properties of this concept with basic relations have been given.

Keywords- Hilbert Space, quasi normal, K-quasinormal, (K-N)-quasinormal,and (K-N)* quasinormal

1-Introduction

Recently many researchers studied the quasinormal operator concepts, such as S. A. Alzuraiqi, A.B. Patel in 2010, [3], introduced n-normal operators on a Hilbert space H. and given some basic properties of these operators also, S. Panayappan and N. Sivamani in 2012,[6], introduced the concept of A-quasinormal operators acting on Hilbert spaces H and in 2015, Salim D. M and Ahmed M.K. [5],introduce another class of normal operator which is (K-N) quasi-normal operator and givensome properties of this concept as well as discussion the relation between this operator with another types of normal operators, but Sivakumar N. andBavithra V.in 2016,[4],introduced the generalization of the above operator called the (K-N) quasi n normal operator and study some basic properties. Finally Eiman H. Abood and Mustafa A. Al-loz in 2016, [2],
introduce some types of generalizations of (n, m) -normal powers operatorsand study some of them properties.
In this paper we given more generalized of quasinormal operator which is (K-N)*-quasinormal operator with some properties of this concept.

2- Some types of quasinormal operator

We recall some types of quasinormal operator with important properties of these concepts as well as relations behind of these concepts, and we start by the definition of quasinormal operator.

Definition (2.1),[3]:-

A bounded linear operator $A: H \longrightarrow H$, *where* H *is Hilbert space then* A *is said to be quasinormal operator if satisfy:-*

$A(A^*A) = (A^*A)A.$

To illustrate that consider the following example.

Example (2.2):-

Let $H = \ell_2(¢) = \{X = (x_1, x_2, x_3, ...) : \sum_{i=1}^{\infty} |x_i|^2 < \infty, x_i \in ¢, \mathrm{i} = 1,2,....\}$ and let U and B the unilateral shift operator and the bilateral shift operator such that:

$U : X \longrightarrow X$ defined by; $U(x_1, x_2, ...) = (0, x_1, x_2, ...)$ $\quad \forall (x_1, x_2, ...) \in \ell_2(¢)$

and $B : X \longrightarrow X$ defined by $B(x_1, x_2, ...) = (x_2, x_3, ...)$ $\quad \forall (x_1, x_2, ...) \in \ell_2(¢)$

Such that $U^* = B$ then U is quasinormal operator.

Proposition (2.3),[3]:-

Let $A \in B(H)$ be quasinormal operator then:-

(1) A^* is quasinormal operator.

(2) A^n is quasinormal operator where n is any positive integer.

(3) $(A^{-1})^*$ is quasinormal operator.

Remark (2.4):-

Let $A, B \in B(H)$ be two quasinormal operators then AB and $A + B$ are not necessary to be quasinormal. To illustrate that, we will take $A = \begin{bmatrix} 0 & -2i \\ 2i & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & i \\ i & i \end{bmatrix}$ are two quasinormal operators on $(¢^2)$ Hilbert space but $A + B = \begin{bmatrix} 0 & -i \\ 3i & i \end{bmatrix}$ is not quasinormal operator, also if we take $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$,

$B=\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ are two quasinormal operators but, $AB=\begin{bmatrix} i & 0 \\ -2i & -i \end{bmatrix}$ is not quasinormal operator.

The following theorem show the necessary condition in order to make remark (2.4) true.

Theorem (2.5):-

If $A, B \in B(H)$ be two quasinormal operators then:-

(1) $A+B$ is quasinormal operator if $AB = BA = A^*B = B^*A = 0$.

(2) AB is quasinormal operator if $AB = BA$ and $AB^* = B^*A$.

Now, the following proposition gives to illustrate the relation between quasinormal operator and (self-adjoint, skew-adjoint, unitary and normal) operator.

Proposition (2.6):-

If $A \in B(H)$ then:-

(1) If A is self-adjoint operator then A is quasinormal operator.

(2)If A is skew-adjointoperator then A is quasinormal operator.

(3) If A is unitary operator then A is quasinormal operator.

(4) If A is normal operator then A is quasinormal operator.

But the converse of above proposition not true in general, to explain that see the following examples.

Examples (2.7):-

(1) Let $A=\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is quasinormal operator but it is clear that it is not self-adjoint operator.

(2)Let $A=\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ is quasinormal operator but it is clear that it is not skew- adjoint operator.

(3) Let $A=\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ is quasinormal operator but it is clear that it is not unitary operator.

(4) Let $H=\ell_2(¢)=\{X=(x_1,x_2,x_3,.....):\sum_{i=1}^{\infty}|x_i|^2<\infty, x_i\in ¢, \mathrm{i}=1,2,....\}$ and let U and B the unilateral and the bilateral shift operators such that ;

$U:X\longrightarrow X$ defined by; $U(x_1,x_2,...)=(0,x_1,x_2,...)$ $\quad \forall(x_1,x_2,...)\in \ell_2(¢)$

and $B:X\longrightarrow X$ defined by $B(x_1,x_2,...)=(x_2,x_3,...)$ $\quad \forall(x_1,x_2,...)\in \ell_2(¢)$

Such that $U^*=B$ then U is quasinormal operator but not normal operator.

Another type of quasinormal has been given by the following definition.

Definition (2.8), [1]:-

A bounded linear operator $A:H\longrightarrow H$ is said to be K-quasinormal operator if satisfy $A(A^*A)^k=(A^*A)^k A$, where k is any positive integer.

To illustrate that consider the following example.

Example (2.9):-

The operator $A=\begin{bmatrix} 0 & 3i \\ -3i & 0 \end{bmatrix}$ on Hilbert space $¢^2$ is K-quasinormal operators if $\mathrm{k}=2$.

Theorem (2.10),[1]:-

Every quasinormal operator is K-quasinormal operator.

Remarks (2.11),[4]:-

(1) When $k=1$ the K-quasinormal operator can to be quasinormal operator.

(2)Let $A, B \in B(H)$ be two K-quasinormal operators then $A+B$ and AB not necessary to be K-quasinormal .To illustrate that, we will take $A = \begin{bmatrix} 0 & 3i \\ -3i & 0 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 \\ 0 & -2i \end{bmatrix}$ on Hilbert space $¢^2$ are K-quasinormal operators I f $\mathrm{k} = 2$

Then $(A+B)(((A+B)^*)(A+B))^2 = \begin{bmatrix} -950+468i & -312+1035i \\ 321-1035i & -468-950i \end{bmatrix}$

but $((A+B)^*(A+B))^2(A+B) = \begin{bmatrix} -950-468i & -312+411i \\ -312-10.1035i & 468-14i \end{bmatrix}$

we get $(A+B)((A+B)^*(A+B))^2 \neq ((A+B)^*(A+B))^2(A+B)$ Therefore $A+B$ is not K-quasinormal operators also, If take $A = \begin{bmatrix} i & 0 \\ 0 & 2i \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 \\ i & -i \end{bmatrix}$ on Hilbert space $¢^2$ are K-quasinormal operators if $\mathrm{k} = 2$ then;

$(AB)((AB)^*(AB))^2 = \begin{bmatrix} -4i & -4i \\ -128 & 128 \end{bmatrix}$, but $((AB)^*(AB))^2(AB) = \begin{bmatrix} 60-34i & -60-34i \\ -68+30i & 68+30i \end{bmatrix}$.

It clears that $(AB)((AB)^*(AB))^2 \neq ((AB)^*(AB))^2(AB)$, therefore AB is not K-quasinormal operator.

Proposition (2.12):-

(1)If $A \in B(H)$ is K-quasinormal operator then A^n is K-quasinormal operator, where n is any positive integer.

(2) If $A \in B(H)$ is K-quasinormal invertible operator then $(A^{-1})^*$ is K-quasinormal operator.

Now, dr.salim and Ahmed [5], introduce generalized of K-quasinormal operator is said to be (K-N)-quasinormal operator by the following definition.

Definition (2.13),[5]:-

A bounded linear operator $A: H \longrightarrow H$ is said to be (K-N)-quasinormal operator if satisfy $A^k(A^*A) = N(A^*A)A^k$, where k is any positive integer and N is bounded operator such that $N: H \longrightarrow H$.

Example (2.14):-

Let $A: \ell^2[0,1] \longrightarrow \ell^2[0,1]$, such that $(Af)(t) = tf(t)$ for all $f \in \ell^2[0,1]$, $t \in \ell[0,1]$ then A is (K-N)-quasinormal operator.

Theorem (2.15),[5]:-

Every K-quasinormal operator is (K-N)-quasinormal operator.

Remarks (2.16),[5]:-

(1) Where $k=1$ with $N=I$ every (K-N)-quasinormal operator can to be quasinormal operator.

(2) Let $A, B \in B(H)$ be two (K-N)-quasinormal operators then $A+B$ and AB not necessary to be (K-N)-quasinormal operators .To illustrate that , we will take $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ 1 & 0 & -i \end{bmatrix}$,are two (K-N) quasinormal operator on $¢^3$ Hilbert space, but $\mathrm{AB} = \begin{bmatrix} 0 & 0 & 2i \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{bmatrix}$ is not (K-N)-quasinormal operator where $k=1$ and $N=I$,also $A+B = \begin{bmatrix} 2 & 0 & -1+i \\ 0 & 2 & 0 \\ 0 & 0 & 1-i \end{bmatrix}$ is not (K-N)-quasinormal operator where $k=1$ and $N=I$.

Proposition (2.17):-

(1) Let $A \in B(H)$ be (K-N)-quasinormal operator then A^n is (K-N)-quasinormal operator where n is any positive integer.

(2) if $A \in B(H)$ is invertible (K-N)-quasinormal operator then $(A^{-1})^*$ is (K-N)-quasinormal operator where $N = (N^{-1})^*$.

Now, the following diagram represents the relation among the above concepts.

$$
\begin{tikzcd}
 & \textit{(K-N)-quasinormal operator} & \\
 & \textit{K-quasinormal operator} \arrow[u] & \\
 & \textit{Quasinormal operator} \arrow[u] & \\
\textit{Normal operator} \arrow[ur] & & \textit{skew-adjoint operator} \arrow[ul] \\
\textit{Self-adjoint operator} \arrow[u] & & \textit{unitary operator} \arrow[ull]
\end{tikzcd}
$$

3-Some properties of (K-N)*-quasinormal operator:

In this section we introduce the definition of (K-N)*-quasinormal operator and illustration this concept via some theorems and examples. We start by the following definition.

Definition (3.1):-

Abounded linear operator $A: H \longrightarrow H$ is said to be (K-N)*-quasinormal operator if satisfy $A^k (A^* A)^k = N(A^* A)^k A^k$ where k is any positive integer and N is bounded operator such that $N: H \longrightarrow H$.

To illustrate this definition consider the following example.

Example (3.2):-

If $A=\begin{bmatrix}2&0&0\\0&5&0\\0&0&9\end{bmatrix}$ *operator on Hilbert space* $¢^3$ *where* $k=2$ *and* $N=I$ *then*

A *is (K-N)*-quasinormal operator.*

Remarks (3.3):-

(1) If A is (K-N)*-quasinormal operator and if $k=1$ with $N=I$ then A is quasinormal operator.

(2) If A is (K-N)*-quasinormal operator and If $(A^*A)^k = A^*A$ then A is (K-N)-quasinormal operator.

(3) If A is (K-N)*-quasinormal operator and If $A^k = A$ and $N=I$ then A is K-quasinormal operator.

Now, we give proposition to show properties of (K-N)*-quasinormal operator.

Proposition (3.4):-

Let $A \in B(H)$ be (K-N)*-quasinormal operator then A^n is (K-N)*-quasinormal operator where n is any positive integer.

Proof:-

Let A be (K-N)*-quasinormal operator, we prove that by using mathematical induction, therefor A is (K-N)*-quasinormal operator, the result is true for $m=1$ that is $A^k(A^*A)^k = N(A^*A)^k A^k$ (i)

we assume the result is true for $m=n$

$(A^k(A^*A)^k)^n = (N(A^*A)^k A^k)^n$ (ii)

to prove the result for $m=n+1$

$(A^k(A^*A)^k)^{n+1} = (A^k(A^*A)^k)^n A^k(A^*A)^k$

$= (N(A^*A)^k A^k)^n (N(A^*A)^k A^k)$ by (i) and (ii),

$= (N(A^*A)^k A^k)^{n+1}$, Thus the result is true for $m = n+1$.

Therefore then A^n is (K-N)*-quasinormal operator.

Remark (3.5):

Let $A \in B(H)$ be (K-N)*quasinormal operator then A^{-1} is not necessary to be (K-N)*-quasinormal operator. To express that, see the following example.

Example (3.6):-

Let $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2i \end{bmatrix}$ be (K-N)*-quasinormal operator if $k = 1$ and $N = I$ but

$A^{-1} = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5i \end{bmatrix}$ is not (K-N)*-quasinormal operator.

Next, to given the condition to make the remark (3.5) true see the following proposition.

Proposition (3.7):-

If $A \in B(H)$ is invertible (K-N)*-quasinormal operator then $(A^{-1})^*$ is (K-N)*-quasinormal operator where $N = (N^{-1})^*$.

Proof:-

Since A *is (K-N)*quasinormal operator then;* $A^k(A^*A)^k = N(A^*A)^k A^k$

By taking inverse and adjoint of both sides we can have;

$((A^{-1})^*)^k (A^{-1}(A^{-1})^*)^k = (N^{-1})^* (A^{-1}(A^{-1})^*)^k ((A^{-1})^*)^k$, *so* $(A^{-1})^*$ *is (K-N)*-quasinormal operator.*

Remark (3.8):-

Let $A, B \in B(H)$ be two (K-N)*-quasinormal operators then $A+B$ and AB not necessary to be true. To show that, considers the following examples.

Examples (3.9):-

(1) Let $A=\begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix}, B=\begin{bmatrix} -1 & 0 \\ 0 & -i \end{bmatrix}$ be two (K-N)*-quasinormal operators on $¢^2$

where $k=2, N=I,$ but $A+B=\begin{bmatrix} -1 & 2i \\ -2i & -i \end{bmatrix}$ is not (K-N)*-quasinormal operator.

(2) Let $A=\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, B=\begin{bmatrix} 0 & i \\ -i & 1 \end{bmatrix}$ be two (K-N)*-quasinormal operators on $¢^2$

where $k=2, N=I.$ but $AB=\begin{bmatrix} 2i & -2 \\ 0 & 2i \end{bmatrix}$ is not (K-N)*-quasinormal operator.

Now, the following theorem given condition to make the remark (3.8) is true.

Theorem (3.10):-

If $A, B \in B(H)$ be two (K-N)*-quasinormal operators then:-

(1) $A+B$ is (K-N)*-quasinormal operator if $AB = BA = A^*B = B^*A = AB^* = BA^* = 0$

(2) AB is (K-N)*-quasinormal operator if $AB = BA$, $BA^* = A^*B, B^*A = AB^*$ and B is quasinormal operator.

Proof:-

(1) $(A+B)^k[(A+B)^*(A+B)]^k = (A^k + B^k)[(A^*)^k A^k + (A^*)^k B^k + (B^*)^k A^k + (B^*)^k B^k]$

$$= A^k (A^*A)^k + B^k (B^*B)^k$$

$$= N(A^*A)^k A^k + N(B^*B)^k B^k$$

$$= N[(A+B)^*(A+B)]^k (A+B)^k$$

Hence $A+B$ is (K-N)*-quasinormal operator.

(2) $(AB)^k[(AB)^*(AB)]^k = A^kB^k[(A^*)^k(B^*)^kA^kB^k]$

$= A^k[(A^*)^kB^kA^k(B^*)^kB^k]$

$= A^k(A^*A)^kB^k(B*B)^k$

$= N[(A^*A)^kA^k(B*B)^kB^k]$

$= N[(A*)^kA^kA^k(B*)^kB^kB^k]$

$= N[(A*)^kA^k(B*)^kA^kB^kB^k]$

$= N[(A^*)^kA^k(B*)^kA^kB^kB^k]$

$= N[(A*)^k(B^*)^kA^kB^kA^kB^k]$

$= N[(AB)^*(AB)]^k(AB)^k$

Hence AB is (K-N)*-quasinormal operator.

Now, the following lemma gives to illustrate the relation between (K-N)*-quasinormal operator and (self adjoint, skew adjoint, normal and unitary) operator.

Lemma (3.11):-

Let $A \in B(H)$ then:-

(1) If A is self-adjoint operator then A is (K-N)*-quasinormal operator.

(2)If A is skew-adjointoperator then A is (K-N)*-quasinormal operator.

(3) If A is normal operator then A is (K-N)*-quasinormal operator.

(4) If A is unitary operator then A is (K-N)*-quasinormal operator.

But the converse of abovelemma in general is not true to explain that see the following example.

Examples (3.12):

(1) Let $A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$; then A is (K-N)*-quasinormal operator on $¢^2$ where $\mathrm{k} = 2,\ \mathrm{N} = \mathrm{I}$ but not self-adjoint operator.

(2) Let $A=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$; then A is (K-N)*-quasinormal operator on $¢^2$ where $\mathrm{k}=2,\ \mathrm{N}=\mathrm{I}$ but not skew-adjoint operator.

(3) Let $H=\ell_2(¢)=\{X=(x_1,x_2,x_3,.....):\sum_{i=1}^{\infty}|x_i|^2<\infty, x_i\in ¢, \mathrm{i}=1,2,....\}$ and let U and B the unilateral and the bilateral shift operators that are defined by:-
$U(x_1,x_2,...)=(0,x_1,x_2,...)\ \forall(x_1,x_2,...)\in\ell_2(¢)$

and $B(x_1,x_2,...)=(x_2,x_3,...)\ \forall(x_1,x_2,...)\in\ell_2(¢)$

Then U is (K-N)*-quasinormal operator where $k=1, N=I$ but not normal.

(4) Let $A=\begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix}$ on Hilbert space $¢^2$ Then A is (K-N)*-quasinormal operator, where $k=2$ and $N=I$ but not unitary.

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