On Modified Three – Step Iteration Process For Three Asymptotically Quasi Nonexpansive Mappings .

Shahla Abd AL-Azeaz Khadum

Directorate – General of Education Baghdad′ s Karkh Third/ Ministry of Education.

E−mail: ouss−kadd@yahoo.com

Abstract:

In the this paper , modified three – step iteration process for three self mappings is introduced in strictly convex and uniformly smooth Banach space . Also the strongly and weakly convergence theorems of this iteration are proved for three asymptotically quasi nonexpansive mappings in uniformly smooth and uniformly convex Banach space.

Keywords: Modified three – step iteration , three asymptotically quasi nonexpansive mappings , Lipschitz single duailty mapping.

حول العمليات التكرارية المحسنة الثلاثية الخطوات لثلاث تطبيقات من نمط شبه الانكماشي المحاذي

م .م شهلاء عبد العزيز كاظم

مديرية تربية بغداد الكرخ الثالثة / وزارة التربية

المستخلص:

في هذا البحث تم تقديم المتتابعات التكرارية المحسنة الثلاثية الخطوات لثلاث تطبيقات في فضاء بناخ المحدب بشدة والمنتظم النعومة وكذلك تم برهنة التقارب القوي والضعيف لهذه المتتابعات لثلاث تطبيقات من نمط شبه الانكماشي المحاذي في فضاء بناخ المنتظم النعومة والتحدب .

1. Introduction

Rafiq [1] introduced three− step iteration process and used it to approximat the unique comman fixed point of three strongly pseudocontractive operators in a real Banach space. Let $E, H, D: G \to G$ are mappings ,for any $x_1 \in G$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\varphi_n)\, x_n + \varphi_n\, E\, y_n$$

$$y_n = (1-\omega_n)\, x_n + \omega_n\, H\, z_n$$

$$z_n = (1-\xi_n)\, x_n + \xi_n\, D\, x_n \quad, \forall n \geq 0 \qquad \ldots(i)$$

$\{\varphi_n\}, \{\omega_n\}$ and $\{\xi_n\}$ are three sequences in [0,1]. Xu and Noor [2] introduced three step− iteration process $\{x_n\}$ as follows, for any $x_1 \in G$

$$x_{n+1} = (1-\varphi_n)\, x_n + \varphi_n\, E^n\, y_n$$

$$y_n = (1-\omega_n)\, x_n + \omega_n\, E^n z_n$$

$$z_n = (1-\xi_n)\, x_n + \xi_n\, E^n x_n \quad, \forall n \geq 1 \qquad \ldots(ii)$$

$\{\varphi_n\}, \{\omega_n\}$ and $\{\xi_n\}$ are real numbers in [0,1] and then who established the convergence results of the modified iteration (ii) in Banach space. Nantadilok [3] presented three− step iteration process with errors and gave condition for convergence of comman fixed point for three generalized asymptotically nonexpansive mappinge.

Let $E, H, D: G \to G$ are mappings ,for any $x_1 \in G$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\varphi_n)\, x_n + \varphi_n\, E^n\, y_n + \mu_n$$

$$y_n = (1-\omega_n)\, x_n + \omega_n\, H^n z_n + \varepsilon_n$$

$$z_n = (1-\xi_n)\, x_n + \xi_n\, D^n x_n + \psi_n \quad, \forall n \geq 1 \qquad \ldots(iii)$$

$\{\mu_n\}, \{\varepsilon_n\}$ and $\{\psi_n\}$ are sequences in K and $\{\varphi_n\}, \{\omega_n\}$ and $\{\xi_n\}$ are real sequencess in [0,1]. If $\xi_n = 0 \; in \; (iii), n \in N$. Then , the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\varphi_n)\, x_n + \varphi_n\, E^n\, y_n + \mu_n$$

$$y_n = (1 - \omega_n)\, x_n + \omega_n\, H^n x_n + \varepsilon_n \;, \forall n \geq 1$$

Is called the Ishikawa iteration sequence.

If $\xi_n = \varphi_n = 0$ in (iii) , $n \in N$. Then , the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \varphi_n)\; x_n + \varphi_n\, E^n\, x_n + \mu_n \;, \forall n \geq 1$$

Is called the Mann iteration sequence.

2. Preliminaries

In this section, some basic definitions and properties concept which needed are presented .

Definition (2.1), $[4]$ and $[5]$:

Let G is a Banach space , K is a nonempty subset of G . A mapping $E: K \to K$ is said to be

(i) Asymptotically nonexpansive .If $\exists$ a sequence $\{e_n\} \subset [1, \infty[$ with $e_n \to 1$ as $n \to \infty$ such that $\|E^n x - E^n y\| \leq e_n \|x - y\|$ ... (1)

$\forall\, x, y \in K$ and $n \geq 1$

(ii)Asymptotically quasi nonexpansive. If $F(E) = \{x \in K : E(x) = x\} \neq \emptyset$ and $\exists$ a sequence $\{e_n\} \subset [1, \infty[$ with $e_n \to 1$ as $n \to \infty$ such that

$$\|E^n x - \rho\| \leq e_n \|x - \rho\| \qquad ... (2)$$

$\forall\, x \in K$ and $n \geq 1$, $\rho \in F$.

Definition (2.2), $[6]$:

Let G is a real Banach space with the dula space G^* and the mapping $J: E \to 2^{E^*}$ is defined by

$J(x) = \{f \in E : \langle x , f \rangle = \|f\| \|x\| , \|f\| = \|x\|\}$, $\forall x \in G$ is said to be normalized duality mapping.

Definition (2.3), [7]:

A Banach space G is said to be (i) Strictly convex . If $\left\|\frac{x+y}{2}\right\| < 1$, for all $x, y \in G$ with $\|x\| = \|y\| = 1$ and $x \neq y$.

(ii) Uniformly convex . If $\lim_{n\to\infty} \|x_n - y_n\| = 1$, for any two sequences $\{x_n\}, \{y_n\}$ $in\ G$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \left\|\frac{x_n+y_n}{2}\right\| = 1$.

Let $U = \{ x \in G : \|x\| = 1\}$ be the unit sphere of G , then the Banach space is said to be

(iii) Smooth . If the $\lim_{n\to\infty} \frac{\|x+ty\|-\|x\|}{2}$ exists , $\forall x, y \in U$.

(iv) Uniformly Smooth . If the limit exists uniformly $\forall x, y \in U$.

Proposition (2.4), [7]:

Let G is a Banach space and $J: G \to 2^{G^*}$ is the normalized duality mapping , then

(i) J $: G \to G^*$ and uniformly norm to norm continuous on each bounded subset of G $\leftrightarrow G$ be a real uniformly smooth Banach space

(ii) $J^{-1} : G^* \to G$, bijective , $JJ^{-1} = I_{G_*}$ and $J^{-1} J = I_G$ if G is reflexive convex and smooth Banach space .

(iii) If G is uniformly smooth , then it is smooth and reflexive . (iv) If G is uniformly convex , then it is strictly convex .

Definition (2.5), [8]

Let G is uniformly smooth Banach space and $B = B_r[0] = \{x \in G ; \|x\| \leq r\}, r > 0$. A duality map $J: B \to G^*$ is said to be $L-$Lipschitz if $\exists\ L > 0$ such that

$$\|J(x) - J(y)\| \leq L \|x - y\| \qquad \dots (3)$$

$\forall\ x, y \in B$.

Definition $(\mathbf{2.6})$, [**9**]:

Let G is a Banach space , K is a nonempty subset of G. A mapping $E: K \to K$ is said to be

(i) Uniformly equi continuous if and only if $\|E^n x_n - E^n y_n\| \to 0$ when ever $\|x_n - y_n\| \to 0$ as $n \to \infty$ for all x_n , $y_n \in G$.

(ii) Semi –compact if ∃ a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \to x \in K$ as $i \to \infty$ for any bounded sequence $\{x_n\}$ in K with $\|x_n - Ex_n\| \to 0$ as $n \to \infty$.

(iii) demiclosed at $y \in G$, if for any sequence $\{x_n\} \in K$ and $x \in G$,

$\{x_n\} \rightharpoonup x$ and $Ex_n \to y$ imply that $x \in K$ and $Ex = y$.

Definition $(\mathbf{2.7})$:

Let G is a strictly convex and uniformly smooth Banach space ,

K is a nonempty and convex subset of G and $E, H, D: K \to K$ are mappings . For any $x_1 \in G$, the iteration sequence $\{x_n\} \subset K$ is defined by

$$x_{n+1} = J^{-1}((1 - \varphi_n) J x_n + \varphi_n J E^n y_n)$$

$$y_n = J^{-1}((1 - \omega_n) J x_n + \omega_n J H^n z_n)$$

$$z_n = J^{-1}((1 - \xi_n) J x_n + \xi_n J D^n x_n), \ \forall n \geq 1 \qquad \dots (4)$$

φ_n , ω_n and ξ_n ara sequence in $[0,1]$. If $\xi_n = 0$ *in* (4) , $\forall n \in N$. Then iteration sequence is defined by

$$x_{n+1} = J^{-1}((1 - \varphi_n) J x_n + \varphi_n J E^n y_n)$$

$$y_n = J^{-1}((1 - \omega_n) J x_n + \omega_n J H^n x_n), \ \forall n \geq 1 \qquad \dots (5)$$

Is called the Ishikawa iteration sequence .

If $\omega_n = 0$ in (5), then iterative sequence defined by

$$x_{n+1} = J^{-1}((1-\varphi_n)Jx_n + \varphi_n JE^n x_n), \forall n \geq 1 \quad ...(6)$$

Is called the Mann iteration sequence.

Lemma (2.8), [6]:

A Banach space G is uniformly convex $\leftrightarrow$ $\exists$ continuous strictly increasing map Υ : $[0,1[\rightarrow [0,1[$ such that

(i) $\Upsilon(0) = 0$

$(ii) \|\tau x + (1-\tau)y\|^2 \leq \tau\|x\|^2 + (1-\tau)\|y\|^2 - \tau(1-\tau)\Upsilon(\|x-y\|)$, $\forall x, y \in B_r[0]$, and $\tau \in [0,1]$.

Lemma (2.9), [9]:

Let G is a uniformly convex Banach space, K is a nonempty convex and bounded subset of G and $E : K \rightarrow K$ is asymptotically nonexpansive mapping, then $I - E$ is demiclosed at 0.

3. Strongly and weakly convergence:

In this section, the following results develop the Main Results of [2], [5], [10], [11] and others.

Theorems (3.1):

Let G is uniformly smooth and uniformly convex, K is a nonempty convex and bounded subset of G. Let $E, H, D: K \rightarrow K$ are quasi asymptotically nonexpansive mappings with $\{e_n\}$, $\{\hbar_n\}$ and $\{d_n\} \subset [1,\infty[$ such that $e_n \rightarrow 1$, $\hbar_n \rightarrow 1$ and $d_n \rightarrow 1$ as $n \rightarrow \infty$. Let J be lipschitz valued mapping defined on K with $L \leq 1$.

Let $\{x_n\}$ define by condition (4) satisfying : $(i) \lim_{n\rightarrow\infty} \varphi_n = \lim_{n\rightarrow\infty} \omega_n = 0$ $(ii) \sum_{n=1}^{\infty} \varphi_n\omega_n < \infty$. *If* $F(E) \cap F(H) \cap F(D) \neq \emptyset$, then

$\| x_n - Ex_n \| \rightarrow 0$, $\|x_n - Hx_n\| \rightarrow 0$ and $\|x_n - Dx_n\| \rightarrow 0$ ...(7)

as $n \rightarrow \infty$.

Proof:

Since K is bounded, there exists $B_r[0]$ for some $r > 0$, from conditions (4) and (2), proposition [(2.4), (ii)], lemma (2.8) and condition (3), $\rho \in F$, then

$$\|z_n-\rho\|^2=\|J^{-1}((1-\xi_n)Jx_n+\xi_n JD^n x_n)-\rho\|^2$$

$$=\left\|J\left(J^{-1}((1-\xi_n)Jx_n+\xi_n JD^n x_n)\right)-J\rho\right\|^2$$

$$=\|(1-\xi_n)Jx_n+\xi_n JD^n x_n-J\rho\|^2$$

$$\leq(1-\xi_n)\|Jx_n-J\rho\|^2+\xi_n\|JD^n x_n-J\rho\|^2$$

$$-\xi_n(1-\xi_n)\Upsilon(\|Jx_n-JD^n x_n\|)$$

$$\leq(1-\xi_n)L^2\|x_n-\rho\|^2+\xi_n L^2\|D^n x_n-\rho\|^2$$

$$-\xi_n(1-\xi_n)\Upsilon(\|Jx_n-JD^n x_n\|)$$

$$\leq(1-\xi_n)\|x_n-\rho\|^2+\xi_n d_n^{\,2}\|x_n-\rho\|^2$$

$$-\xi_n(1-\xi_n)\Upsilon(\|Jx_n-JD^n x_n\|)\qquad\ldots(8)$$

From condition (4), (2), proposition [(2.4) , (ii)], lemma (2.8) and condition (3), $\rho\in F$, then

$$\|y_n-\rho\|^2=\|J^{-1}((1-\omega_n)Jx_n+\omega_n JH^n z_n)-\rho\|^2$$

$$=\|J(J^{-1}((1-\omega_n)Jx_n+\omega_n JH^n z_n))-J\rho\|^2$$

$$=\|(1-\omega_n)Jx_n+\omega_n JH^n z_n-J\rho\|^2$$

$$\leq(1-\omega_n)\|Jx_n-J\rho\|^2+\omega_n\|JH^n z_n-J\rho\|^2$$

$$-\omega_n(1-\omega_n)\Upsilon(\|Jx_n-JH^n z_n\|)$$

$$\leq(1-\omega_n)L^2\|x_n-\rho\|^2+\omega_n L^2\|H^n z_n-\rho\|^2$$

$$-\omega_n(1-\omega_n)\Upsilon(\|Jx_n-JH^n z_n\|)$$

$$\leq(1-\omega_n)\|x_n-\rho\|^2+\omega_n \hbar_n^{\,2}\|z_n-\rho\|^2$$

$$-\omega_n(1-\omega_n)\Upsilon(\|Jx_n-JH^n z_n\|)\qquad\ldots(9)$$

Putting (8) into (9), leads to

$$\|y_n-\rho\|^2\leq(1-\omega_n)\|x_n-\rho\|^2+\omega_n\hbar_n^{\,2}(1-\xi_n)\|x_n-\rho\|^2$$

$$+\omega_n\ ħ_n{}^2\ \xi_n\ ȡ_n{}^2\ \|\ x_n\ -\rho\|^2\ -\omega_n\ ħ_n{}^2\ \xi_n(1-\ \xi_n)$$

$$\Upsilon(\|\ Jx_n - JD^n x_n\|)$$

$$-\omega_n\,(1-\ \omega_n)\ \Upsilon(\|\ Jx_n - JH^n ẕ_n\|) \qquad \ldots(10)$$

From condition (4) and (2), proposition [(2.4) , (ii)], lemma (2.8) and condition (3), $\rho \in F$. Then

$$\|x_{n+1} - \rho\ \|^2 = \|J^{-1}\,(\,(1-\ \varphi_n)\,Jx_n +\ \varphi_n\,JE^n y_n\ \ ) - \rho\ \ \|^2$$

$$= \left\|J\big(J^{-1}\,(\,(1-\ \varphi_n)\,Jx_n +\ \varphi_n\,JE^n y_n\ \ )\big) - J\rho\right\|^2$$

$$= \|(1-\ \varphi_n)\,Jx_n +\ \varphi_n\,JE^n y_n\ \ - J\rho\|^2$$

$$\leq (1-\ \varphi_n)\|\ Jx_n - J\rho\|^2\ + \varphi_n\,\|JE^n y_n\ \ - J\rho\|^2$$

$$-\varphi_n\,(1-\ \varphi_n)\ \Upsilon(\|\ Jx_n - JE^n y_n\|)$$

$$\leq (1-\ \varphi_n)\ Ł^2\,\|\ x_n - \rho\|^2\ +\ \varphi_n\ Ł^2\,\|\ E^n y_n\ \ - \rho\|^2$$

$$-\varphi_n\,(1-\ \varphi_n)\ \Upsilon(\|\ Jx_n - JE^n y_n\|)$$

$$\leq (1-\ \varphi_n)\ \|\ x_n - \rho\|^2\ +\ \varphi_n\ ẹ_n{}^2\|\ y_n\ \ - \rho\|^2$$

$$-\varphi_n\,(1-\ \varphi_n)\ \Upsilon(\|\ Jx_n - JE^n y_n\|) \qquad \ldots(11)$$

Putting (10) into (11), imply that

$$\|x_{n+1} - \rho\ \|^2$$

$$\leq [(1-\ \varphi_n) + \varphi_n ẹ_n{}^2(1-\ \omega_n) + \varphi_n ẹ_n{}^2\ \omega_n ħ_n{}^2\ (1-\ \xi_n)$$

$$+\ \varphi_n ẹ_n{}^2\ \omega_n ħ_n{}^2 \xi_n ȡ_n{}^2]\ \|\ x_n - \rho\|^2$$

$$-\ \varphi_n ẹ_n{}^2\ \omega_n ħ_n{}^2 \xi_n\ (1-\ \xi_n)\ \Upsilon(\|\ Jx_n - JD^n x_n\|)$$

$$-\ \varphi_n ẹ_n{}^2\ \omega_n(1-\ \omega_n)\ \Upsilon(\|\ J\chi_n - JH^n ẕ_n\|)$$

$$-\ \varphi_n\ (1-\ \varphi_n)\Upsilon\,(\|\ Jx_n - JE^n y_n\|)$$

$$\leq [(1-\ \varphi_n)\kappa_n{}^8 + \varphi_n \kappa_n{}^8(1-\ \omega_n) + \varphi_n\ \omega_n \kappa_n{}^8\ (1-\ \xi_n)$$

$$+\ \varphi_n\ \omega_n\ \xi_n \kappa_n{}^8\ ]\|\ x_n - \rho\|^2 -\ \varphi_n ẹ_n{}^2\ \omega_n ħ_n{}^2 \xi_n\ (1-\ \xi_n)$$

$$\Upsilon(\| Jx_n - JD^n x_n\|) - \varphi_n \mathrm{e}_n{}^2 \omega_n (1-\omega_n)\Upsilon(\| Jx_n - JH^n z_n\|)$$

$$-\varphi_n (1-\varphi_n)\Upsilon(\| Jx_n - JE^n y_n\|)$$

$$\leq \| x_n - \rho\|^2 - \varphi_n \mathrm{e}_n{}^2 \omega_n \hbar_n{}^2 \xi_n (1-\xi_n)\ \Upsilon(\| Jx_n - JD^n x_n\|)$$

$$-\varphi_n \mathrm{e}_n{}^2 \omega_n (1-\omega_n)\Upsilon(\| Jx_n - JH^n z_n\|)$$

$$-\varphi_n (1-\varphi_n)\Upsilon(\| Jx_n - JE^n y_n\|) \quad \ldots(12)$$

From (12), three inequalities be obtained . That are

$$\|x_{n+1} - \rho\|^2 \leq$$

$$\| x_n - \rho\|^2 - \varphi_n \mathrm{e}_n{}^2 \omega_n \hbar_n{}^2 \xi_n (1-\xi_n)\ \Upsilon(\| Jx_n - JD^n x_n\|) \quad \ldots(13)$$

$$\|x_{n+1} - \rho\|^2 \leq \| x_n - \rho\|^2 - \varphi_n \mathrm{e}_n{}^2 \omega_n (1-\omega_n)\Upsilon(\| Jx_n - JH^n z_n\|) \quad (14)$$

$$\|x_{n+1} - \rho\|^2 \leq \| x_n - \rho\|^2 - \varphi_n (1-\varphi_n)\Upsilon(\| Jx_n - JE^n y_n\|) \quad \ldots(15)$$

Suppose that $\exists\, n_0 \in N \ni \{\| x_n - \rho\|\}$ be decreasing and bounded then $\{\| x_n - \rho\|\}$ convergent .

Then $\lim_{n\to\infty}\{\| x_n - \rho\| - \| x_{n+1} - \rho\|\} = 0$ (16)

By using (16), $\sum_{n=1}^{\infty} \varphi_n \omega_n < \infty$ and $\lim_{n\to\infty} \xi_n = 0$ *in* (13)

Then , $\lim_{n\to\infty} \Upsilon(\| Jx_n - JD^n x_n\|) = 0.$

There fore $\lim_{n\to\infty} (\| Jx_n - JD^n x_n\|) = 0$.

From proposition [(2.4) (i)] , then $\lim_{n\to\infty} \| x_n - D^n x_n\| = 0$ ... (17)

Simillary by use (16) and $\sum_{n=1}^{\infty} \varphi_n \omega_n < \infty$ in (14), lead to

$\lim_{n\to\infty} \Upsilon(\| Jx_n - JH^n z_n\|) = 0.$

There fore $\lim_{n\to\infty} (\| Jx_n - JH^n z_n\|) = 0$.

Also , from proposition [(2.4) (i)] , then

$\lim_{n\to\infty} (\| x_n - H^n z_n \|) = 0$ ... (18)

Also ,by use (16) and $\lim_{n\to\infty} \varphi_n = 0$ in (15),

Then $\lim_{n\to\infty} \Upsilon(\| Jx_n - JE^n y_n \|) = 0.$

There fore $\lim_{n\to\infty} (\| Jx_n - JE^n y_n \|) = 0.$

Also , from proposition [(2.4) (i)], lead to

$\lim_{n\to\infty} (\| x_n - E^n y_n \|) = 0.$ ... (19)

From conditions (4) and (17), imply that

$\|x_n - z_n\| = \xi_n \|x_n - D^n x_n\| \to 0 \quad as \quad n \to \infty$ ... (20)

Using conditons (18), (1) and (20), then

$\|x_n - H^n x_n\| \leq \|x_n - H^n z_n\| + \| H^n z_n - H^n x_n\|$

$\leq \|x_n - H^n z_n\| + \hbar_n \| z_n - x_n\| \to 0 \quad as \quad n \to \infty$ ... (21)

From condition (4) and (18) , lead to

$\|x_n - y_n\| = \omega_n \| x_n - H^n z_n\| \to 0 \quad as \quad n \to \infty$ ... (22)

By use conditions (19), (1) and (22) , then

$\|x_n - E^n x_n\| \leq \|x_n - E^n y_n\| + \| E^n y_n - E^n x_n\|$

$\leq \|x_n - E^n y_n\| + e_n \| y_n - x_n\| \to 0 \quad as \quad n \to \infty$ (23)

By condition (4) and (19), which imply that

$\|x_n - x_{n+1}\| = \varphi_n \| x_n - E^n y_n\| \to 0 \quad as \quad n \to \infty$ ... (24)

From conditions (24), (23) and (19), then

$\|x_n - Ex_n\| \leq \|x_n - x_{n+1}\| + \| x_{n+1} - E^{n+1} x_{n+1}\|$

$$+\|E^{n+1}x_{n+1}-E^{n+1}x_n\|+\|E^{n+1}x_n-Ex_n\|$$

$$\leq \|x_n-x_{n+1}\|+\|x_{n+1}-E^{n+1}x_{n+1}\|$$

$$+\mathrm{e}_{n+1}\ \|x_{n+1}-x_n\|+\mathrm{e}_n\|E^n x_n-x_n\|$$

$$\rightarrow 0 \quad as\ n\ \rightarrow\ \infty \qquad \ldots(25)$$

Similary ,from $(24), (22)$ and condition (1), lead to

$\|x_n-Hx_n\|\ \rightarrow\ 0 \quad as\ n\ \rightarrow \infty$

Similary ,from $(24), (17)$ and condition (1), then

$\|x_n-Dx_n\|\ \rightarrow\ 0 \quad as\ n\ \rightarrow\ \infty$

Introduce main theorems by using theorem (3.1).

Theorem $(\mathbf{3.2})$:

Let G and K as in the theorem (3.1) and $E,H,D\colon K\rightarrow K$ are uniformly equi continuous mapping and semi compact and quasi asymptotically nonexpansive mappings with$\{e_n\}$, $\{h_n\}$ and $\{d_n\}\subset [1,\infty[$ such that $e_n\rightarrow 1$, $\hbar_n\rightarrow 1$ and $d_n\rightarrow 1$ as $n\rightarrow\infty$. Let J be lipschitz singule valued map defined on K with $L\leq 1$. Let $\{x_n\}$ define as in (4) satisfying:

$(i)\ \lim\limits_{n\rightarrow\infty}\varphi_n=0\ and\ \lim\limits_{n\rightarrow\infty}\omega_n=0 \quad (ii)\ \sum_{n=1}^{\infty}\varphi_n\omega_n<\infty$.

$If F=F(E)\cap F(H)\cap F(D)\ \neq\emptyset$, $then\ \lim\limits_{n\rightarrow\infty}\|x_n-\rho\|=0$, $\rho\in F$.

Proof:

Let H is a semi compact . As $\lim\limits_{n\rightarrow\infty}\|x_n-Hx_n\|=0$, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightarrow 0$ $as\ i\rightarrow\infty$. Using $x_n=x_{n_i}$ in (7), Then $\lim\limits_{n\rightarrow\infty}\|x_{n_i}-Hx_{n_i}\|=0$ and the uniformly equi continuous of mapping H .we get $\rho=H\rho$ and so $\rho\in F$. Similary , we prove that $\rho=E\rho$ and so $\rho\in F$ and $\rho=D\rho$ and so $\rho\in F$.

Corollary(3.3.)

Let G , K and $E,H,:K \to K$ as in the theorem (3.2). Also E is uniformly equi continuous mappinq and J be lipschitz singule valued map defined on k with $L \leq 1$. Let $\{x_n\}$ define by condition (5) satisfying (i) $\lim_{n\to\infty} \varphi_n = 0$ and $\lim_{n\to\infty} \omega_n = 0$ (ii) $\sum_{n=1}^{\infty} \varphi_n \omega_n < \infty$. If $Ғ = Ғ(E) \cap Ғ(H) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - \rho\| = 0 , \rho \in Ғ$.

Proof:

Follows from theorems(3.2) and (3.1) with $\xi_n = 0$, $\forall\, n \geq 1$.

Corollory (3.4.):

Let G , K and $E{:}\,K \to K$ as in the theorem (3.2). Also E is uniformly equi continuous mappinq and J be lipschitz singule valued map defined on K with $L \leq 1$. Let $\{x_n\}$ define by condition (6) with $\lim_{n\to\infty} \varphi_n = 0$ and $\sum_{n=1}^{\infty} \varphi_n < \infty$. If $Ғ = Ғ(E) \neq \emptyset$, then

$\lim_{n\to\infty} \|x_n - \rho\| = 0 , \rho \in Ғ$.

Proof:

Follows from theorems (3.2) and (3.1) with $\xi_n = \omega_n = 0$, $\forall\, n \geq 1$

Theorem (3.5)

Let G and K as in the theorem (3.1) , $K$is closed and $E,H,D{:}\,K \to K$ is quasi asymptotically nonexpansive mappings with $\{ɐ_n\}$, $\{ħ_n\}$ and $\{ȡ_n\} \subset [1,\infty[$ such that $\lim_{n\to\infty} ɐ_n = 1$, $\lim_{n\to\infty} ħ_n = 1$ and $\lim_{n\to\infty} ȡ_n = 1$. Let J be lipschitz singule valued map defined on K with $L \leq 1$. Let$\{x_n\}$ define by condition (3) satisfyin : $(i) \lim_{n\to\infty} \varphi_n = 0 \; and \; \lim_{n\to\infty} \omega_n = 0$.

(ii) $\sum_{n=1}^{\infty} \varphi_n \omega_n < \infty$. If $Ғ = Ғ\,(E) \cap Ғ\,(H) \cap Ғ\,(D) \neq \emptyset$, then $x_n \rightharpoonup \rho$,

$\rho \in Ғ$.

Proof:

Let $W_w(x_n) = \{ y \in K{:}\, x_{n_k} \rightharpoonup y \;\, for \{x_{n_k}\} \subseteq \{x_n\} \}$ where $W_w(x_n)$

is the weak $w-$ limit set of $\{x_n\}$. since K be closed and bounded, $\exists\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \rightharpoonup \rho \in W_w(x_n) \neq \emptyset$ as $i \to \infty$ and by theorem (3.1) and by using $x_n = x_{n_i}$, then $\lim_{n\to\infty}\|x_{n_i} - Ex_{n_i}\| = 0$ $\lim_{n\to\infty}\|x_{n_i} - Hx_{n_i}\| = 0$ and $\lim_{n\to\infty}\|\chi_{n_i} - Dx_{n_i}\| = 0$, since $I-E$, $I-H$ and $I-D$ are demiclosed at 0 by lemma (2.9), therefore $E\rho = \rho$, $H\rho = \rho$ and $D\rho = \rho$. That is $W_w(\chi_n) \subset$ Ғ.

Corollary(3.6)

Let G, K and $E, H, : K \to K$ as in the theorem(3.5) .Let J be lipschitz singule valued map defined on K with $L \le 1$. Let $\{x_n\}$ defined by condition (5) with (i) $\lim_{n\to\infty} \varphi_n = 0$ *and* $\lim_{n\to\infty} \omega_n = 0$ (ii) $\sum_{n=1}^{\infty} \varphi_n \omega_n < \infty$.

If Ғ $=$ Ғ$(E) \cap$ Ғ$(H) \neq \emptyset$, *then* *then* $x_n \rightharpoonup \rho$, $\rho \in$ Ғ.

Proof:

Follows from theorems (3.5) and(3.1) with $\xi_n = 0$, $\forall\, n \ge 1$.

Corollory (3.7):

Let G, K and $E: K \to K$ as in the theorem (3.5). Let J is lipschitz singule valued map defined on K with $L \le 1$. Let $\{x_n\}$ define as in (6) with $\lim_{n\to\infty} \varphi_n = 0$ and $\sum_{n=1}^{\infty} \varphi_n < \infty$. If Ғ $=$ Ғ$(E) \neq \emptyset$, then $x_n \rightharpoonup \rho$, $\rho \in$ Ғ.

Proof:

Follows from theorems (3.5) and(3.1) with $\xi_n = \omega_n = 0$, $\forall\, n \ge 1$

4. Conclusions:

Let G is uniformly smooth and uniformly convex, then Modified three – iteration process convergence strongly to comman fixed of three semi compact and quasi asymptotically nonexpansive self mappings while Modified three – iteration process convergence weakly to comman fixed of three quasi asymptotically nonexpansive self mappings and $I-E, I-H$, $I-D$ are demiclosed at 0.

References:

[1] Rafiq A. , ″Modified Noor Iteration for Nonlinear Equations in Banach Spaces″ ,App. M. and Com.,Vol(182),pp(589-595),2006.

[2] Xu B.and Noor M.A.,″Fixed-point Iterations for Asymptotically Nonexpansive Mappings in Banach space″ ,J.Math.Anal,Appl.Vol(267),pp(444-453),2002.

[3] Nantadilok J.,″Three-Step Iteration Scheme with Errors for Generalized Asymptotically Quasi Nonexpansive Mappings″ ,Thai J. Math .Vol(6), pp(297-308),2008.

[4] Goebel K.and Kirk W.A., ″A fixed point for Asymptotically Nonexpansive Mappings″. Proc. Amer. Math. Soc., vol.(35), pp(171-174), 1972.

[5] Qin X.,Su Y.and Shang M.,″Strongly Convergence for Three Classes of Uiformly Equi-Continuous and Asymptotically Quasi-Nonexpansive Mappings″,J.Korean Math.Soc.Vol.(45),No.I,pp.(29-40),2008.

[6] Zhu.J.,Change S. and Liu M.,″Strong Convergernce theorems by Halpern-Mann Iteration for Muli-valued Relatively Nonexpansive Mappings in Banach space with Applications″,J. of Inequalities and Applications,2010

[7] Okeke G.A and Olaleru J.O.,″Modified Noor Iterations with Errors for Nonlinear Equation in Banach Spaces″ ,J. of Nonlinear Sci.Appl.,vol.(7).

[8]Chidume C.E.and Moore C.,″Fixed Point Iteration for Pseudocontractive Maps″,1997.

[9] Thianwan T.,″Weak Convergence of the Projection Type Ishikawa Iteration Scheme for Two Asyptotically Nonexpansive Nonself Mappings″,2015

[10] Khan S.H.and Takahashi W.,″Approximating Comman Fixed Points for Two Asymptotically Nonexpansive Mappings″,Sci.Math. Jpn., Vol.(53),No.(1),pp.(143-148),2001.

[11]Schu J.,″Weak and Strongly convergence to fixed points Asymptotically Nonexpansive Mappings″, Bull, Austal. Math. Soc.,Vol(43), No.(1), pp(154-159),1991.