Iterative Sequences With Errors For Quasi Uniformly Hölder Continuous Self (and Nonself) Mappings.

Shahla Abd AL-Azeaz Khadum

Directorate General of Education Baghdad s Karkh Third

shhhhhh71@yahoo.com

. **Abstract**.

In this paper , iterative sequences with errors are introduced and strongly convergence theorems of thems are established when A, L self (and nonself) quasi uniformly Hölder continuous mappings for a real Banach space Z and for a nonempty convex and bounded subset K of Z. Our results improve and generalize the results Khan and Takahashi,Saluja and many others.

Keywords. Iterative sequences with errors ,uniformly Hölder continuous mappings, quasi uniformly Hölder continuous mappings and Banach space

المتتابعات التكرارية مع الاخطاء للتطبقين الذاتيين(والغيرالذاتيين) من نمط شبه هولدر المنتظم الاستمرارية .

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شهلاء عبد العزيز كاظم

المديرية العامة لتربية بغداد الكرخ الثالثة

الملخص

في هذا البحث ،يتم تقديم متتابعات تكرارية مع الاخطاء وأنشأت نظريات التقارب القوي عندما يكون كل من التطبيقين الذاتيين (والغير ذاتيين) من نمط شبه هولدر المنتظم الاستمرارية في فضاء بناخ عندما تكون المجموعة الجزئية منه غير خالية ومحدبة ومقيدة .نتائجنا تحسين وتعميم نتائج كل من خان وتاكاهاشي ،سالوجا وغيرها الكثير.

1 −**Introduction**

In 1991, Schu [7] introdused the iterative sequence $\{x_n\}$ is a nonempty and convex subset K of Hilbert space H as, for given $x_1 \in K$

$$y_n = (1 - b_n)x_n + b_n A^n x_n , \forall n \geq 1$$

$$x_{n+1} = (1 - a_n)x_n + a_n A^n y_n , \forall n \geq 1 \qquad ...(i)$$

where $\{b_n\}$ and $\{a_n\}$ be sequences in$(0,1)$ and A be a self mapping . Who proved weak and strongly convergence theorems of the iterative sequence which he defined by (i) for asymptotically nonexpansive self mapping A in Hilbert space$H$. Liu[4] pressented the iterative sequence$\{x_n\}$ with errors as ,for given $x_1 \in K$

$$y_n = a_n x_n + b_n A^n x_n + c_n u_n , \forall n \geq 1$$

$$x_{n+1} = d_n x_n + e_n A^n y_n + h_n v_n , \forall n \geq 1 \quad ...(ii)$$

where$\{u_n\}$ and $\{v_n\}$ be bounded sequences and$\{a_n\}\{b_n\}\{c_n\}\{d_n\}$ $\{e_n\}$and$\{r_n\}$ are sequence in $[0,1]$ with $a_n + b_n + c_n = 1$ and $d_n + e_n + h_n = 1$.Who introduced strong convergence results of iterative sequence which he defined by (ii) for quasi asymptotically nonexpansive self mapping in a uniformly convex Banach space Z. Cho, Guo and Zhou [3] extand and improved theorems of iterative sequence (ii) to uniformly $\varphi -$continuous and asymptotically quasi nonexpansive in a uniformly convex Banach space$Z$. In 2001,Khan andTakahashi [3] introdused and studied the iterative sequence $\{x_n\}$ as ,for given x_1 in K.

$$y_n = (1 - b_n)x_n + b_n A^n x_n , \forall n \geq 1$$

$$x_{n+1} = (1 - a_n)x_n + a_n L^n y_n , \forall n \geq 1 \qquad ...(iii)$$

Where $A\ and\ L$ are asymptotically nonexpansive self mappings on K with $\{a_n\}and\{b_n\} \in (0,1)$. In 2010 ,Khan and Din [2] established convergence results of iterative sequence which they defined by (iii) with errors for two uniformly equi-continuous self mappings $A\ and\ L$ in a uniformly convex Banach space Z. In 2014 Saluja [6] generalized the iterative sequence $\{x_n\}$ with errors as , for $x_1 \in K \subseteq Z$

$$y_n = P(a_n x_n + b_n A_1 (PA_1)^{n-1} x_n + c_n v_n), \forall n \geq 1$$

$$x_{n+1} = P(d_n A_1 (PA_1)^{n-1} x_n + e_n A_2 (PA_2)^{n-1} y_n + r_n u_n), \forall n \geq 1 \;\; \ldots (iv)$$

Where P is the nonexpansive retraction of Z onto K ,$\{v_n\}\{u_n\}$ be bounded sequences in K, $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} and \{e_n\} \in$ [0,1] with $a_n + b_n + c_n = 1 = d_n + e_n + r_n$ and proved weak convergence theorems of the iterative sequence defined by (iv) for two asymptotically nonexpansive nonself mappings in uniformly convex Banach space Z.

2- Preliminaries

Definition (2.1) : [1] and [6]

Let Z be a real Banach space and K be a nonempty subset of Z. Amapping $A: K \to K$ $(or\ A: K \to Z)$ is said to be

(i) Uniformly Hölder continuous .If ∃ constants $S \geq 0$ and $r \in (0,1)$
$\ni \forall\ x, y\ in\ K$ and $n \geq 1$

$$\|A^n x_n - A^n y_n\| \leq S\, \|x_n - y_n\|^r \quad \ldots (1)$$

or

$$\|A(PA)^{n-1} x_n - A(PA)^{n-1} y_n\| \leq S\, \|x_n - y_n\|^r \quad \ldots (2)$$

Where P is a nonexpansive retraction of $Z\ onto\ K$.

(ii) Quasi uniformly Hölder continuous. If F(A) $= \{x \in K: A(x) = x\} \neq \emptyset$ and ∃ constants $S \geq 0$ and $r \in (0,1)$

$$\|A^n x_n - q\| \leq S\|x_n - q\|^r \quad \ldots (3)$$

Or $$\|A(PA)^{n-1} x_n - q\| \leq S\, \|x_n - q\|^r \quad \ldots (4)$$
where $q \in F(A)$

Example (2.2):

Amapping $A: R \to R$ (where $R =$ real number with usuall norm) such that $A(x) = -\frac{1}{3} \cos x$ is uniformly Hölder contiuous with $S = \frac{1}{3}$ and $r = 1$.

When the boundness of the derivative for A .Then it is uniformly Hölder continous i.e.

$$|A'_X| = \left|\frac{1}{3}\sin x\right| \le \frac{1}{3}$$

So $|A_X - A_Y| \le \frac{1}{3}|x-y| \ \forall x, y \ in\ Z\ and\ r = 1$

Defintion (2.3) :

Let Z be a real Banach space, K a nonempty and convex subset of Z and $A, L: K \to K$ .For any $x_1\ in\ K$, the iterative sequence $\{x_n\} \in K$ is defined by

$$y_n = (1-b_n)x_n + b_n A^n x_n + d_n w_n, \forall n \ge 1$$

$$x_{n+1} = (1-a_n)L^n x_n + a_n L^n y_n + c_n v_n, \forall n \ge 1 \quad \ldots (5)$$

Where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be bounded sequences in K.

$If\ A, L: K \to Z\ and\ P: Z \to K$, then the iterative sequence $\{x_n\}$ in (5) is defined by

$$y_n = P\big((1-b_n)x_n + b_n A(PA)^{n-1}x_n + d_n w_n\big), \forall n \ge 1$$

$$x_{n+1} = P\big((1-a_n)L(PL)^{n-1}x_n + a_n L(PL)^{n-1}y_n + c_n v_n\big), \forall n \ge 1$$

$$\ldots (6)$$

where P is the nonexpansive retraction of $Z\ onto\ K$

Lemma (2.4): [7]

Suppose that $\{\mu_n\}, \{\xi_n\}$ and $\{t_n\}$ are real sequence in $[1, \infty[$

satisfying

$$\mu_n = (1+\xi_n)\mu_n + t_n, \forall n \ge 1$$

If $\sum_{n=1}^{\infty} \xi_n < \infty, \sum_{n=1}^{\infty} t_n < \infty, then\ \lim_{n\to\infty} \mu_n\ exist.$ In particular ,if $\{\mu_n\}$ has a subsequence $\mu_{n_i} \ni \mu_{n_i} \to 0\ as\ n \to \infty$, then $\lim_{n\to\infty} \mu_n = 0$.

Lemma (2.5): [8]

Let Z be a Banach space, $\{x_n\}$ and $\{y_n\}$ are bounded sequences in Z and $\{\beta_n\}$ is a sequence in $[0,1]$ with

$$0 < \lim_{n\to\infty} inf\beta_n \leq \lim_{n\to\infty} sup\beta_n < 1$$

Suppose that

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n \,, \forall n \geq 0$$

And $\lim_{n\to\infty} \sup(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$.

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$

3-Main theorems

Our purpose in this paper to prove that the iterative sequences with errors for two quasi uniformly Hölder continuous self (and nonself) mappings strongly convergence to a comman fixed points. The results presented in this paper generalize the corresponding Main Results in[3], [6] and others.

Theorem (3.1) :

Let Z be a real Banach space, K *be* a nonempty convex and bounded subset of Z and $A, L: K \to K$ be quasi uniformly Hölder continuous mappings with constants $S_1, S_2 \leq 1$ *and* $r_1, r_2 \in [0,1]$. Let $\{x_n\}$ defined by condition (5) with real sequences $\{a_n\}, \{b_n\}, \{c_n\}$ *and* $\{d_n\} \in [0,1]$ satisfying

$,\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} d_n < \infty, \sum_{n=1}^{\infty} a_n\, b_n < \infty, a\ subsequence\ a_{n_i}$

$of\ a_n \ni a_{n_i} \to 0\ as\ n \to \infty\,, h \leq a_{n_i} \leq (1-h)\ for\ some\ h > 0$

$and\ \{w_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}\ be$ bounded sequences in K.Then

$If\ F = F(A) \cap F(L) \neq \emptyset, we\ have\ \lim_{n\to\infty}\|x_n - p\| = 0\ for\ p \in F.$

Proof: let $P \in F$,from condition (5) and condition (3) ,we get

$$\|y_n - p\| = \|(1-b_n)x_n + b_n A^n x_n + d_n w_n - p\|$$

$$\leq (1-b_n)\|x_n - \mathrm{p}\| + b_n\|A^n x_n - p\| + d_n\|w_n - p\|$$

$$\leq (1-b_n)\|x_n - p\| + b_n R_1\|x_n - p\|^{r_1} + B_n$$

$$\leq (1-b_n)\|x_n - p\| + b_n R_1\|x_n - p\| + B_n \qquad ...(7)$$

Where $B_n = d_n\|w_n - p\|$.Since $\sum_{n=1}^{\infty} d_n < \infty, then\ \sum_{n=1}^{\infty} B_n < \infty$.

Again from conditions (5), (3) and condition (7) ,we obtion

$$\|x_{n+1} - p\| = \|(1-a_n)L^n x_n + a_n L^n y_n + c_n v_n - p\|$$

$$\le (1-a_n)\|L^n x_n - p\| + a_n\|L^n y_n - p\| + c_n\|v_n - p\|$$

$$\le (1-a_n)S_2\|x_n - p\|^{r_2} + a_n S_2\|y_n - p\|^{r_2} + G_n$$

$$\le (1-a_n)S_2\|x_n - p\| + a_n S_2\|y_n - p\| + G_n$$

where $G_n = c_n\|v_n - P\|$ .Since $\sum_{n=1}^{\infty} c_n < \infty, then\ \sum_{n=1}^{\infty} G_n < \infty$.

Suppose that $S = max\{S_1, S_2\}$

$$\|x_{n+1} - \mathrm{p}\| \le (1-\mathrm{a_n})S\|x_n - p\| + \mathrm{a_n}S\{(1-b_n)\|x_n - p\| + b_n S\|x_n - p\| + B_n\} + G_n$$

$$\le (S + a_n b_n S^2)\|x_n - p\| + a_n S B_n + G_n$$

$$\le (1 + a_n b_n S^2)\|x_n - \mathrm{p}\| + \mathrm{H_n}$$

Where $H_n = a_n S B_n + G_n$. Since $\sum_{n=1}^{\infty} \mathrm{B_n} < \infty\ and \sum_{n=1}^{\infty} \mathrm{G_n} < \infty$,

Then $\sum_{n=1}^{\infty} H_n < \infty$.

Let us denote $\qquad ɰ_n = \|x_n - p\|$

$$\xi_n = a_n b_n S^2\ ,\sum_{n=1}^{\infty} \xi n < \infty; \forall n \ge 1,$$

and using lemma (2.4) ,we obtion $\lim_{n\to\infty} M_n$ exists. Which imiplies that $\lim_{n\to\infty}\|x_n - p\|$ exists. Suppose that $\{\|\mathrm{x_{n_i}} - \mathrm{p}\|\}$ is a subsequence of $\{\|x_n - p\|\}$ and a_{n_i} is a subsequence of a_n ,since a_{n_i} satisfying $h \le a_{n_i} \le (1-h)$,we get

$$\{\|\mathrm{x_{n_i}} - \mathrm{p}\|\} = (1 - \mathrm{a_{n_i}})\{\|\mathrm{x_{n_i}} - \mathrm{p}\|\} + \mathrm{a_{n_i}}\{\|\mathrm{x_{n_i}} - \mathrm{p}\|\}$$

$$\le h\{\|\mathrm{x_{n_i}} - \mathrm{p}\|\} + a_{\mathrm{n_i}}\{\|\mathrm{x_{n_i}} - \mathrm{p}\|\}$$

$$\le 2\mathrm{a_{n_i}}D\ , where\ D = sup\{\|\mathrm{x_{n_i}} - \mathrm{p}\|\}$$

since $a_{n_i} \to 0\ as\ n \to \infty, then\ \lim_{n\to\infty}\{\|\mathrm{x_{n_i}} - \mathrm{p}\|\} = 0$

Again using lemma (2.4) ,we get $\lim_{n\to\infty} ɰ_n = 0$, so that

$\lim_{n\to\infty}\|x_n - p\| = 0$ . ... (8)

Now ,by using theorm (3.1) and lemma (2.5),we obtain the next theorem

Theorem (3.2):

Let Z be a real Banach space,K be a nonempty convex and bounded subset of Z and mappings $A, L: K \to K$ are quasi uniformly Hölder continuous with constants $S_1, S_2 \le 1$ and $r_1, r_2 \in (0,1)$ .Let $\{x_n\}$ defined by condition (5) with real sequences $\{a_n\}, \{b_n\}, \{c_n\}$ $and\{d_n\} \in [0,1]$ $\ni \sum_1^\infty c_n < \infty$, $\sum_{n=1}^\infty d_n < \infty$ and $\sum_{n=1}^\infty an\, bn < \infty$, a subsequence a_{n_i} of a_n $\ni a_{n_i} \to 0$ as $n \to \infty$ and $h \le a_{n_i} \le (1-h)$ $for\ some\ h > 0$ and $\{v_n\}_1^\infty$,

$\{w_n\}_1^\infty be\ bounded\ sequence\ in\ K$. $If\ F = F(A) \cap \mathrm{F(L)} \neq \emptyset\ and$

$0 < \lim_{n\to\infty} inf\ a_n \le \lim_{n\to\infty} sup\ a_n < 1$, $0 < \lim_{n\to\infty} inf b_n \le \lim_{n\to\infty} sup\ b_n < 1$.

Then $\lim_{n\to\infty}\|x_n - A^n x_n\| = 0 = \lim_{n\to\infty}\|x_n - L^n x_n\|$

Proof: from conditions (1) $and(3)$,we obtion

$$\|A^{n+1}x_n - A^n x_n\| \le \|A^{n+1}x_{n+1} - A^{n+1}x_n\| + \|A^{n+1}x_n - p\|$$

$$+\|p - A^n x_n\|$$

$$\le S_1\|x_{n+1} - x_n\|^{r_1} + S_1\|x_n - p\|^{r_1} + S_1\|p - x_n\|^{r_1}$$

$$\le \|x_{n+1} - x_n\| + 2\|x_n - p\| \qquad ... (9)$$

From conditions $(8) and (9)$, we get

$$\lim_{n\to\infty} sup(\|A^{n+1}x_{n+1} - A^n x_n\| - \|x_{n+1} - x_n\|) \le 0$$

Then, by lemma $(2.5), we\ get \lim_{n\to\infty}\|x_n - A^n x_n\| = 0$ ... (10)

From conditions(1) and (3),we get

$$\|L^{n+1}y_n - L^n y_n\| \le \|L^{n+1}y_{n+1} - L^{n+1}y_n\| + \|L^{n+1}y_n - p\|$$

$$+\|p - L^n y_n\|$$

$$\le R_2\|y_{n+1} - y_n\|^{r_2} + R_2\|y_n - p\|^{r_2} + R_2\|p - y_n\|^{r_2}$$

$$\le \|y_{n+1} - y_n\| + 2\|y_n - p\| \qquad ... (11)$$

$$\|y_{n+1} - y_n\| \le \|x_{n+1} - x_n\| + b_{n+1}\|A^{n+1}x_{n+1} - x_{n+1}\|$$

$+ \ b_n\|x_n - A^n x_n\| + d_{n+1}\|w_{n+1}\| - d_n\|w_n\| \quad ...(12)$

From conditions (8) and (10), $we\ get$
$\|A^n x_n - p\| \leq \|A^n x_n - x_n\| + \|x_n - p\| \to 0\ as\ n \to \infty, hence$
$\|y_n - p\| = (1 - b_n)\|x_n - p\| + b_n\|A^n x_n - p\| + d_{n+1}\|w_n - p\|$
$\to 0\ as\ \ n \to \infty \quad ...(13)$

$\|L^{n+1}y_{n+1} - L^n y_n\| \leq \|x_{n+1} - x_n\| + b_{n+1}\|A^{n+1}x_{n+1} - x_{n+1}\| +$
$b_n\|x_n - A^n x_n\| + 2\|y_n - p\| + d_{n+1}\|w_{n+1}\| - d_n\|w_n\| \quad ...(14)$

from conditions(1) and (3),we get

$$\|L^{n+1}x_{n+1} - L^n x_n\| = \|L^{n+1}x_{n+1} - L^{n+1}x_n\| + \|L^{n+1}x_n - p\| + \|p - L^n x_n\|$$

$$\leq R_2\|x_{n+1} - x_n\|^{r_2} + R_2\|x_n - p\|^{r_2} + R_2\|x_n - p\|^{r_2}$$
$$\leq \|x_{n+1} - x_n\| + 2\|x_n - p\| \quad ...(15)$$

From conditions (14) , (10), (13)and(15) ,we get
$\lim_{n\to\infty} sup(\|L^{n+1}y_{n+1} - L^n y_n\| - \|L^{n+1}x_{n+1} - L^n x_n\|) \leq 0$

Hence ,from lemma (2.5), we get

$$\|L^n x_n - L^n y_n\| \to 0\ as\ n \to \infty \quad ...(16)$$

From conditions (8) , (3) and (13),we get

$$\|x_n - L^n y_n\| \leq \|x_n - p\| + \|L^n y_n - p\|$$

$$\leq \|x_n - p\| + R_2\|y_n - p\|^{r_2}$$

$$\leq \|x_n - p\| + \|y_n - p\| \ \to 0\ as\ n \to \infty \quad ...(17)$$

from conditions (17) and (16),we get

$$\|x_n - L^n x_n\| \leq \|x_n - L^n y_n\| + \|L^n y_n - L^n x_n\|$$

$$\to 0\ as\ \ n \to \infty \quad ...(18)$$

Now , we present next theorem
Theorem (3.3) :

Let Z be a real Banach space, K be a nonempty convex and bounded subset of Z and $\mathrm{A, L: K \to K}$ are quasi uniformly Hölder continuous mappings with constants S_1, S_2 and $r_1, r_2 \in (0,1)$. Let $\{x_n\}$ defined by condition (5) with real sequences $\{a_n\}, \{b_n\}$, $\{c_n\}$ and $\{d_n\} \in [0,1]$with $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} a_n b_n < \infty$, $\{v_n\}_1^{\infty}$ *and* $\{w_n\}_1^{\infty}$ are bounded sequences *in* K, *a supsequence* a_{n_i} *of*

a_n *and* $a_{n_i} \to 0$ *as* $n \to \infty$ *with* $h \leq a_{n_i} \leq (1-h)$.

$If F = F(A) \cap F(L) \neq \emptyset$, $\lim\limits_{n\to\infty} \|x_n - A^n x_n\| = 0 = \lim\limits_{n\to\infty} \|x_n - L^n x_n\|$

$lim_{n\to\infty} \|x_n - p\| = 0, P \in F$ Then $P is$ a comman fixed point of A and L.

Proof :by triangle inequalite , conditions (3), (8) *and* (10), *we get*

$$\|p - A_P\| \leq \|p - x_n\| + \|x_n - A^n x_n\| + \|A^n x_n - A_P\|$$

$$\leq \|p - x_n\| + \|x_n - A^n x_n\| + R_1 \|A^{n-1} x_n - p\|^r$$

$$\leq \|p - x_n\| + \|x_n - A^n x_n\| + \| A^{n-1} x_n - p\|$$

$$\leq \|p - x_n\| + \|x_n - A^n x_n\| + R_1 \| x_n - p\|^r$$

Taking the limit as $n \to \infty$, we get that $p = A_P$, similary by using condition(7) ,we get $p = L_p$, *there for* $A_P = p = L_P$

In the next theorem , we use the retraction mapping P where P: Z → K

Theorem(3.4):

Let Z be a real Banach space,K be a nonempty, convex and bounded subset of Z *and* $A, L: Z \to K$ are quasi uniformly Hölder continuous mappings with constants $E_1, E_2 \leq 1$ *and* $e_1, e_2 \in (0,1)$ .Let $\{x_n\}$ defined by condition (6) with real sequences $\{a_n\}, \{b_n\}, \{c_n\}$*and* $\{d_n\} \in [0,1]$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} a_n b_n < \infty$ *and* a subsequence a_{n_i} *of* a_n ∋ $a_{n_i} \to 0$ *as* $n \to \infty$ *and* $h \leq a_{n_i} \leq (1-h)$ *for some* $h > 0$ *and* $\{v_n\}_1^{\infty}, \{w_n\}_1^{\infty}$ be bounded sequences in K .
$If\ F = F(A) \cap F(L) \neq \emptyset, \lim_{n\to\infty} \|x_n - q\| = 0$, *for any point* $q \in F$.

Proof : Let $q \in F$,from conditions (6) , P is a nonexpansive mapping and condition(4), we get

$$\|y_n - q\| = \| P\big((1-b_n)x_n + b_n A(PA)^{n-1} x_n + d_n w_n - P(q)\big)\|$$

$$\leq \|(1-b_n)x_n + b_n A(PA)^{n-1} x_n + d_n w_n - q\|$$

$$\leq (1-b_n)\|x_n - q\| + b_n \|A(PA)^{n-1} x_n - q\| + d_n \|w_n - q\|$$

$\leq (1-b_n)\|x_n - q\| + b_n E_1 \|x_n - q\|^{e_1} + d_n H_n$

$\leq (1-b_n)\|x_n - q\| + b_n E_1 \|x_n - q\| + d_n H_n$ ... (19)

Where $H_n = d_n\|w_n - q\|. Since \sum_{n=1}^{\infty} d_n < \infty , then \sum_{n=1}^{\infty} H_n < \infty$. Again from condition (6), P is a nonexpansive mapping and condition

(19),we obtain

$\|x_{n+1} - q\| = \|P((1-a_n)L(PL)^{n-1}x_n + a_n L(PL)^{n-1}y_n + c_n v_n - P(q))\|$

$\leq \|(1-a_n)L(PL)^{n-1}x_n + a_n L(PL)^{n-1}y_n + c_n v_n - q\|$

$\leq (1-a_n)\|L(PL)^{n-1}x_n - q\| + a_n\|L(PL)^{n-1}y_n - q\| + c_n\|v_n - q\|$

$\leq (1-a_n)E_2\|x_n - q\|^{e_2} + a_n E_2\|y_n - q\|^{e_2} + c_n\|v_n - q\|$

$\leq (1-a_n)E_2\|x_n - q\| + a_n E_2\|y_n - q\| + U_n$

Where $U_n = c_n\|v_n - q\|. Since \sum_{n=1}^{\infty} c_n < \infty, then \sum_{n=1}^{\infty} U_n < \infty$

Suppose that $E = max\{E_1, E_2\}$

$\|x_{n+1} - q\| \leq (1-a_n)E\|x_n - q\| + a_n E\{(1-b_n)\|x_n - q\| + b_n E\|x_n - q\| + H_n\} + U_n$

$\leq (E + a_n b_n E^2)\|x_n - q\| + a_n E H_n + U_n$

$\leq (1 + a_n b_n E^2)\|x_n - q\| + t_n$

$Where\ t_n = a_n E H_n + U_n, since \sum_1^{\infty} c_n < \infty, then \sum_1^{\infty} U_n < \infty$

Let us denote

$$M_n = \|x_n - q\|$$

$$\xi_n = a_n b_n E^2, \forall n \geq 1$$

And using lemma (2.4), $we\ get \lim_{n\to\infty} M_n\ exists$. Which implies that

$\lim_{n\to\infty}\|x_n - q\|$ exists , using the same the proof of theorem(3.1),we get $\lim_{n\to\infty}\|x_n - q\| = 0$.

By using theorem (3.4) ,we get the next theorem

Theorem (3.5) :

Let Z be a real Banach space, K be a nonempty convex and bounded subset of Z $and\ A, L : Z \to K$ be quasi uniformly Hölder continuous mappings with constants $E_1, E_2 \leq 1\ and, e_1, e_2 \in (0,1)$. Let $\{x_n\}$ defined by condition (6) with real sequences $\{a_n\}, \{b_n\}, \{c_n\}$ $and\ \{d_n\} \in [0,1]$,$\sum_{n=1}^{\infty} cn < \infty, \sum_{n=1}^{\infty} dn < \infty\ and\ \sum_{n=1}^{\infty} an\ bn\ < \infty$.

A subsequence $a_{n_i}\ of a_n \ni a_{n_i} \to 0\ as\ n \to \infty\ and\ h \leq a_{n_i} \leq (1-h)$ $for\ some\ h > 0\ and\ \{v_n\}_1^{\infty}, \{w_n\}_1^{\infty}\ be$ bounded sequence in Z. $If 0 < \lim_{n\to\infty} inf a_n \leq$ $\lim_{n\to\infty} sup a_n < 1, 0 < \lim_{n\to\infty} inf b_n \leq \lim_{n\to\infty} sup b_n < 1\ and\ F = F(A) \cap F(L) \neq \emptyset . Then$

$$\lim_{n\to\infty} \|x_n - A^n x_n\| = 0 = \lim_{n\to\infty} \|x_n - L^n x_n\|$$

Proof: Since P is a nonexpansive mapping and condition(2),we get

$$\|P(A(PA)^n x_{n+1}) - P(A(PA)^{n-1} x_n)\| \leq \|A(PA)^n x_{n+1} - A(PA)^{n-1} x_n\|$$

$$\leq \|A(PA)^n x_{n+1} - A(PA)^n x_n\|$$

$$+\|A(PA)^n x_n - q\| + \|q - A(PA)^{n-1} x_n\|$$

$$\leq E_1 \|x_{n+1} - x_n\|^{e_1} + E_1 \|x_n - q\|^{e_1} + E_1 \|q - x_n\|^{e_1}$$

$$\leq \|x_{n+1} - x_n\| + 2\|x_n - q\| \qquad \ldots (21)$$

$$\|Px_{n+1} - Px_n\| \leq \|x_{n+1} - x_n\| \qquad \ldots (22)$$

From conditions (20), $(21) and (22)$,we get

$$\lim_{n\to\infty} sup(\|P(A(PA)^n x_{n+1}) - P(A(PA)^{n-1} x_n)\| - \|Px_{n+1} - Px_n\|) \leq 0$$

Then by lemma(2.5), $we\ get \|P(x_n) - P(A(PA)^{n-1} x_n)\| \to 0\ as\ n \to \infty$

and P is a nonexpansive mapping .

$That\ is\ \|x_n - A(PA)^{n-1} x_n\| \to \infty\ as\ n \to \infty \quad \ldots (23)$

From the nonexpansive mapping P and conditions $(2) and (4)$.

We get

$$\|P(L(PL)^n y_{n+1}) - P(L(PL)^{n-1} y_n)\|$$
$$\leq \|P(L(PL)^n y_{n+1}) - P(L(PL)^n y_n)\| + \|P(L(PL)^n y_n) - P(q)\| + \|P(q) - L(PL)^{n-1} y_n\|$$

$$\leq \|L(PL)^n y_{n+1} - L(PL)^n y_n\| + \|L(PL)^n y_n - q\| + \|q - L(PL)^{n-1} y_n\|$$

$$\leq E_2\|y_{n+1} - y_n\|^{e_2} + E_2\|y_n - q\|^{e_2} + E_2\|q - y_n\|^{e_2}$$

$$\leq \|y_{n+1} - y_n\| + 2\|y_n - q\| \qquad \dots (24)$$

$$\|y_{n+1} - y_n\| \leq \|Px_{n+1} - Px_n\| + b_{n+1}\|P(A(PA)^n x_{n+1}) - Px_{n+1}\| +$$

$$b_n\|Px_n - P(A(PA)^{n-1} x_n)\| + d_{n+1}\|w_{n+1}\| - d_n\|w_n\| \qquad \dots (25)$$

$$\leq \|x_{n+1} - x_n\| + b_{n+1}\|A(PA)^n x_{n+1} - x_{n+1}\| +$$

$$b_n\|x_n - A(PA)^{n-1} x_n\| \qquad \dots (26)$$

From conditions (20) and (23) ,we get

$$\|P(A(PA)^{n-1})x_n - q\| \leq \|A(PA)^{n-1} x_n - q\|$$

$\leq \|A(PA)^{n-1} x_n - x_n\| + \|x_n - q\| \to 0 \ as \ n \to \infty$,hence

$$\|y_n - q\| = (1 - b_n)\|Px_n - q\| + b_n\|P(A(PA)^{n-1} x_n - q)\| + d_n\|w_n - q\|$$

$$\leq (1 - b_n)\|x_n - q\| + b_n\|A(PA)^{n-1} x_n - q\| +$$

$$d_n\|w_n - q\| \to 0 \ as \ n \to \infty \qquad \dots (27)$$

Butting (27) into (26) ,we get

$$\|P(L(PL)^n y_{n+1}) - P(L(PL)^n y_n)\| \leq \|x_{n+1} - x_n\|$$

$$+b_{n+1}\|A(PA)^n x_{n+1} - x_{n+1}\| + b_n\|x_n - A(PA)^{n-1} x_n\| + 2\|y_n - q\| + d_{n+1}\|w_{n+1}\| - d_n\|w_n\| \qquad \dots (28)$$

Since P is a nonexpansive mapping and conditions (2) and(4).

We get

$$\|P(L(PL)^n x_{n+1}) - P(L(PL)^{n-1} x_n)\|$$
$$\leq \|P(L(PL)^n x_{n+1}) - P(L(PL)^n x_n)\|$$
$$+ \|P(L(PL)^n x_n) - P(q)\| + \|P(q) + P(L(PL)^{n-1} x_n)\|$$

$\leq \|L(PL)^n x_{n+1} - L(PL)^n x_n\| + \|L(PL)^n x_n - q\| + \|q - L(PL)^{n-1} x_n\|$

$\leq E_2\|x_{n+1} - x_n\|^{e_2} + E_2\|x_n - q\|^{e_2} + E_2\|q - x_n\|^{e_2}$

$\leq \|x_{n+1} - x_n\| + 2\|x_n - q\|$ ...(29)

From conditions (28), (23), (27) and(29),we get

$$\lim_{n\to\infty} \sup(\|P(L(PL)^n y_{n+1}) - P(L(PL)^{n-1} y_n)\| - \|P(L(PL)^n x_{n+1}) - P(L(PL)^{n-1} x_n)\|) \leq 0$$

Hence from lemma (2.5), $we\ get$ $\|P(L(PL)^{n-1} y_n) - P(L(PL)^{n-1} x_n)\| \to 0\ as\ n \to \infty$

That is $\|L(PL)^{n-1} y_n - L(PL)^{n-1} x_n\| \to 0\ as\ n \to \infty$ ...(30)

From conditions(20), (4)and(28),we get

$$\|x_n - L(PL)^{n-1} y_n\| \leq \|x_n - q\| + \|q - L(PL)^{n-1} y_n\|$$

$$\leq \|x_n - \mathrm{q}\| - \mathrm{E}_2\|\mathrm{q} - \mathrm{y}_\mathrm{n}\|^{\mathrm{e}_2}$$

$$\|x_n - q\| - \|y_n - q\| \to 0\ as\ n \to \infty \qquad ...(31)$$

From conditions (31)and(30),we get

$$\|x_n - L(PL)^{n-1} x_n\| \leq \|x_n - L(PL)^{n-1} y_n\| + \|L(PL)^{n-1} y_n - L(PL)^{n-1} x_n\| \to 0\ as\ n \to \infty \qquad ...(32)$$

Theorem (3.6) :

Let Z be a real Banach space,K be a nonempty convex and bounded subset of Z and $A, L : Z \to K$ are quasi uniformly Hölder continuous mappings with $E_1, E_2 \leq 1\ and\ e_1, e_2 \in (0,1)$.Let $\{x_n\}$ defined by condition (6) with real sequences $\{a_n\}, \{b_n\}, \{c_n\}\ and\ \{d_n\} \in [0,1]$,$\sum_{,n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} d_n < \infty\ and\ \sum_{n=1}^{\infty} a_n b_n < \infty, \{v_n\}_1^{\infty}$ $and\ \{w_n\}_1^{\infty}$and{w_n} be bounded sequences in Z,a subsequence a_{n_i} of $a_n \ni a_{n_i} \to 0\ as\ n \to \infty\ and\ h \leq a_{n_i} \leq (1-h)\ for\ some\ h > o$ and $0 < \lim_{n\to\infty} inf a_n \leq \lim_{n\to\infty} sup a_n < 1$, $0 < \lim_{n\to\infty} inf b_n \leq \lim_{n\to\infty} sup b_n < 1$. $If F(A) \cap F(L) \neq \emptyset$, $\lim_{n\to\infty}\|x_n - A^n x_n\| = 0\ and\ \lim_{n\to\infty}\|x_n - L^n x_n\| = 0,$

$then$ $\lim_{n\to\infty}\|x_n - q\| = 0$, q is a comman fixed point $A\ and\ L$.

Proof: Let $q \in F$,by conditions $(4), (20), (23) and (32)$ and using the same proof of the theorem (3.3),we get $A_q = q = L_q$

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