On Some Types of S-Connected Spaces

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Abstract:-

In this work, we introduce and study new types of S-connected spaces, namely (T-S-connected, T^*-S-connected and ST-S-connected) spaces, where T is an operator associated with the topology τ on a non-empty set X, and some new types of (T, L)-continuous functions. Several properties of these spaces and functions are proved.

المستخلص:-

في هذا العمل سنقوم بتقديم دراسة أنواع جديدة من الفضاءات شبه المتصلة تدعى [الفضاء شبه المتصل-T، الفضاء شبه المتصل-T^*، الفضاء شبه المتصل-ST)، وبعض أنواع جديدة من الدوال المستمرة (T, L)، وبعض الصفات حول تلك الدوال والفضاءات سوف تبرهن.

1. Introduction

In 1963, N. Levin (N. Levin, 1963) introduced a new class of open sets is called semi-open sets in topology y and use these sets to study semi-continuity in topological spaces, see [4].

The concepts [operator topological space, T-open set, T^*-open set, (T, L)-continuous function and T-connected space] was introduced by Hadi J. and Ali in 2004 [1], Hadi J. and S. Al-Kuttibi in [2] and

In this work, we introduce and study new class S-connected spaces namely (T-S-connected space, T^*-S-connected space and ST-S-connected space), T is an operator associated with the topology τ on a non-empty set X, and define new class of (T, L)-continuous function types are ((T, L)-S-continuous function, (T^*, L)-S-continuous function and (ST, L)-S-continuous functions), and find the relation between these functions. Also we will study continuous image of these functions with respect to (T-S-connected space, T^*-S-connected and ST-S-connected space).

2. Some Basic Concepts

Some definition and basic concepts have been recalled in this section.

Definition (2-1),[5]:- A subset A of a topological space X is said to be **semi-open** if $A \subseteq CL(\text{int}(A))$ and denoted by **S-open**.

Remark (2-2),[5]:- In any topological space, it is clear that every open set is S-open, but the converse is not in general. To illustrate that consider the following example.

Example (2-3):- Let R be the real line with the usual topology, and A = (a, b], where a, b $\in$ R and a < b, then A is S-open but is not open.

Remark (2-4),[5] [6]:- In any topological space X.

1. So(x) will denote the class of all S-open subsets of X.
2. The union of any collection of S-open subsets of X is S-open in X.
3. The intersection of two S-open sets in X is not S-open in general.

Definition (2-5),[6]:- A space X is called **connected space** if it is not the union of two disjoint non-empty open sets, otherwise is called **disconnected space**.

Definition (2-6),[4]:- A space X is said to be **S-disconnected space** if for each two non-empty S-open sets A, B in X, then:

1. $X = A \cup B$.
2. $A \cap B = \phi$.

Definition (2-7),[4]:- A space X is said to be **S-connected space** if X is not disconnected.

Remark (2-8),[4]:- Since every (open) set is (S-open), then every S-connected space is connected, and we explain that by the following diagram:

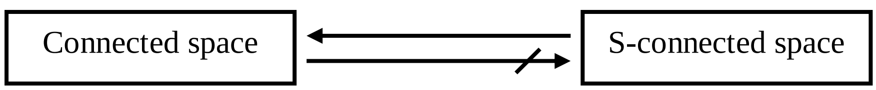

Diagram (1)

Definition (2-9) [1]:- Let (X, τ) be a topological space and T: P(X) → P(X) be a function such that $W \subseteq T(W)$, $W \in \tau$, then we say that T is an operator associated with the topology τ on X, and the triple (X, τ, T) is called an **operator topological space** and denoted by (O.T.S).

Definition (2-10) [1]:- Let (X, τ, T) be an operator topological space and K $\subseteq$ K, then K is said to be **T-open set**, if for each $x \in K$, there exists $G \in \tau$ such that, $x \in G \subseteq T(G) \subseteq K$.So every T-open set is open.

Example (2-11):- Let (X, τ, T) be a topological space and let T: P(X) → P(X) be a function defined as follows: T(A) = cL int A. If A is an open set in X, then A ⊆ cL int A = T(A), then T is an operator associated with the topology τ on X and the triple (X, τ, T) be an operator topological space.

Now, if A is not open and satisfies A ⊆ T(A) = cL int A, then A is called S-open.

Definition (2-12),[3]:- Let (X, τ, T) be an operator topological space X is said to be T-connected if it is not the union of two disjoint non-empty T-open subsets of X.

Definition (2-13),[3]:- Let (X, τ, T) → (Y, σ, L) be a function, then f is said to be (T, L)-continuous function if the inverse of every L-open in Y is T-open in X.

Theorem (2-14),[3]:- Let f: (X, τ, T) → (Y, σ, L) be a (T, L)-continuous function. If X is T-connected, then Y is L-connected.

Now we introduce new class of open set in O.T.S by the following definition.

Definition (2-15):-Let(X, τ, T) be an operator topological space and A ⊆ X, then A is called:

1. T-S-open set. If A is S-open set and A ⊆ T(A).
2. T^*-S-open set. If ∀ x ∈ A there exists S-open set G in X such that x∈G⊆int T(G) ⊆ A.
3. ST-S-open set. If ∀ x ∈ A there exists S-open set G in X such that , x ∈ G ⊆ int T(G) ⊆ A.

From above definition, we can get the following implications. It shows the relation between these types of T-S-open sets.

$$T^*\text{-S-open} \rightarrow \text{ST-S-open} \rightarrow \text{S-open} \rightarrow \text{T-S-open}$$

Next, we introduce new class of S-connected space upon the class of T-S-open sets.

Definition (2-16) [3] :- Let (X, τ, T) be an operator topological space, we say:

1. X is T-S-disconnected if it is the union of two disjoint non-empty T-S-open subsets of X. Otherwise, X is called T-S-connected space.
2. X is T^*-S-disconnected if it is the union of two disjoint non-empty T^*-S-open subsets of X. Otherwise, X is called T^*-S-connected space.
3. X is ST-S-disconnected if it is the union of two disjoint non-empty ST-S-open subsets of X. Otherwise, X is called ST-S-connected space.

From above definitions, we can get the following diagram. It shows the relation between these types of S-connected spaces.

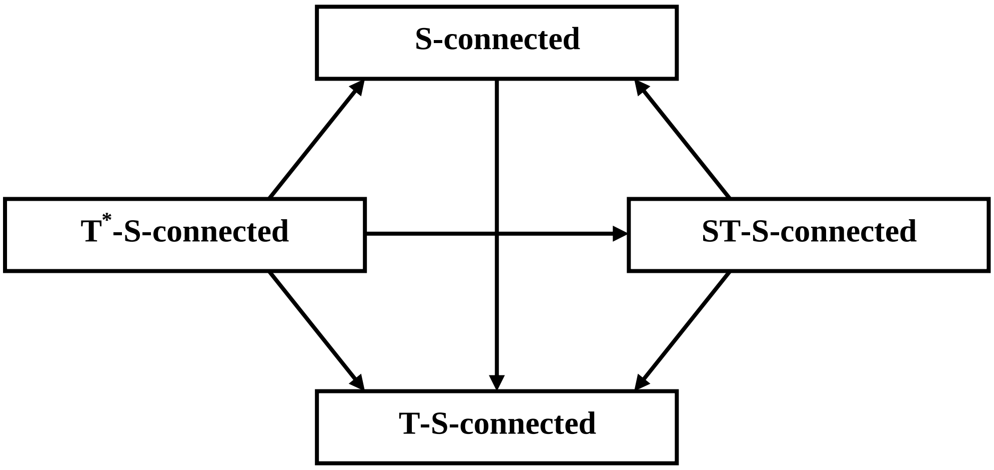

Diagram (2)

Now we introduce some new types of (T, L)-continuous functions:

Definition (2-17):-

1. (T, L)-S-continuous: if the inverse of every L-S-open set in Y is an T-S-open in X.
2. (T^*, L)-S-continuous: if the inverse of every L-S-open set in Y is an T^*-S-open in X.
3. (ST, L)-S-continuous: if the inverse of every L-S-open set in Y is an ST-S-open in X.

3. Main Results

In this section we study and discuss the relation between ((T, L)-S-continuous function, (T^*, L)-S-continuous function, (ST, L)-S-continuous function), and we will study continuous image of these functions with respect to (T-S-connected, T^*-S-connected and ST-S-connected) spaces.

Proposition (3-1):- Every (T^*, L)-S-continuous function is (T, L)-S-continuous.

Proof:-

Let f: $(X, \tau, T) \rightarrow (Y, \sigma, L)$ be an (T^*, L)-S-continuous. T-P f is (T, L)-S-continuous function. Let A be an L-S-open set in Y. Since f is (T^*, L)-S-continuous

Then, $f^{-1}(A)$ is an T^*-S-open set in X, by definition (2-15) (every T^*-S-open is T-S-open) we get , $f^{-1}(A)$ is an T-S-open in X. Thus f is (T, L)-S-continuous function.

Proposition (3-2):- Every (T^*, L)-S-continuous function is (ST, L)-S-continuous.

Proof:-

Let f: (X, τ, T) → (Y, σ, L) be an (T^*, L)-S-continuous, and A be an L-S-open set in Y. Since f is (T^*, L)-S-continuous function .Then, f^{-1}(A) is an T^*-S-open set in X

By definition (2-15) (every T^*-S-open is ST-S-open) we get, f^{-1}(A) is an ST-S-open in X. Thus f is (ST, L)-S-continuous function.

Proposition (3-3):- Every (ST, L)-S-continuous function is (T, L)-S-continuous.

Proof:-

Let f: (X, τ, T) → (Y, σ, L) be an (ST, L)-S-continuous function, and A be an L-S-open set in Y. Since f is (ST, L)-S-continuous. Then, f^{-1}(A) is an ST-S-open set in X

By definition (2-15) (every ST-S-open is T-S-open) we get, f^{-1}(A) is an T-S-open in X

Thus f is (T, L)-S-continuous function.

The following diagram shows the relation between (T, L)-continuous types.

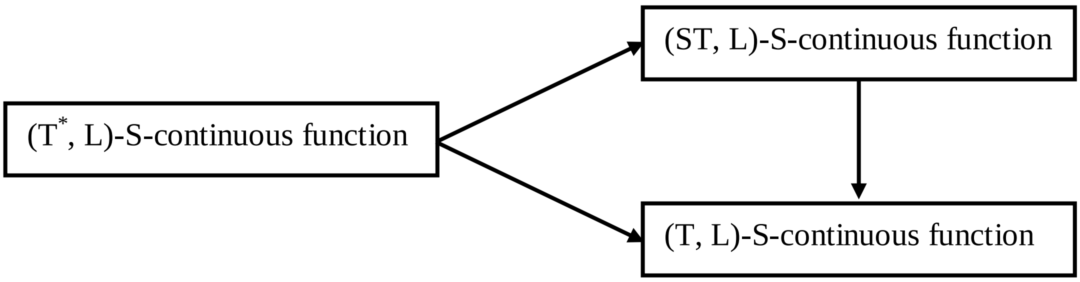

Diagram (3)

Summarized the last proposition where the converse is not necessary true.

Now, we prove several proposition about the image of these functions types w.r.s (T-S-connected, T^*-S-connected and ST-S-connected) spaces**.**

Proposition (3-4):- If f: (X, τ, T) → (Y, σ, L) is a (T, L)-S-continuous onto function and if X is T-S-connected, then Y is L^*-S-connected space.

Proof:-

Suppose Y is L^*-S-disconnected space. Then, there exist two disjoint non-empty L^*-S-open sets A, B in Y such that $Y = A \cup B$, by definition (2-15) $\Rightarrow$ (every L^*-S-open is L-S-open) $\Rightarrow$ A and B are also L-S-open sets in Y. Since f is (T, L)-S-continuous onto function.Then, $f^{-1}(A)$, $f^{-1}(B)$ are two T-S-open sets in X such that

$X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A) \cap f^{-1}(B) = \phi$

Thus, X is T-S-disconnected space (which are contradiction).Since X is T-S-connected space. Hence, Y must be L^*-S-connected space

Proposition (3-5):- If f: $(X, \tau, T) \rightarrow (Y, \sigma, L)$ is a (T, L)-S-continuous onto function and if X is T-S-connected, then Y is ST-S-connected space.

Proof:-

Suppose Y is ST-S-disconnected space. Then, there exist two disjoint non-empty ST-S-open sets A, B in Y such that $Y = A \cup B$,by definition (2-15) $\Rightarrow$ (every SL-S-open is L-S-open) $\Rightarrow$ A and B are also L-S-open sets in Y .Since f is (T, L)-S-continuous onto function .Then, $f^{-1}(A)$, $f^{-1}(B)$ are two T-S-open sets in X such that,

$X = f^{-1}(Y) = f^{-1}(A \cup B = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A) \cap f^{-1}(B) = \phi$

Thus, X is T-S-disconnected space (which are contradiction)

Since X is T-S-connected space. Hence, Y must be ST-S-connected space

Similarly, we prove the following proposition**:**

Proposition (3-6):- If f: $(X, \tau, T) \rightarrow (Y, \sigma, L)$ is a (T, L)-S-continuous onto function and if X is T-S-connected, then Y is L-S-connected space.

Corollary (3-7):- If f: $(X, \tau, T) \rightarrow (Y, \sigma, L)$ be (ST, L)-S-continuous onto function and if X is T-S-connected, then Y is [L^*-S-connected space, SL-S-connected space, L-S-connected space] respectively.

Proof:-

This follows from Proposition (3-3), then f is (T, L)-S-continuous, and by using Propositions (3-4), (3-5) and (3-6) we have Y is (L^*-S-connected, SL-S-connected and L-S-connected) spaces respectively.

Corollary (3-8):- If f: (X, τ, T) → (Y, σ, L) be (T^*, L)-S-continuous onto function and if X is T-S-connected, then Y is [L^*-S-connected space, SL-S-connected space, L-S-connected space] respectively.

Proof:- This follows from Proposition (3-2) and Corollary (3-7).

Proposition (3-9):- If f: (X, τ, T) → (Y, σ, L) be (ST, L)-S-continuous onto function and if X is ST-S-connected space, then Y is L^*-S-connected space.

Proof:-

Suppose Y is L-S-disconnected space. Then, there exist two disjoint non-empty L^*-S-open sets A, B in Y such that Y = A ∪ B, by definition (2-15) ⇒ (every L^*-S-open is L-S-open) ⇒ A and B are also L-S-open sets in Y.Since f is (ST, L)-S-continuous onto function. Then, $f^{-1}(A)$, $f^{-1}(B)$ are two ST-S-open sets in X such that

$X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A) \cap f^{-1}(B) = \phi$

Thus, X is ST-S-disconnected space (which are contradiction)Since X is ST-S-connected space. Hence, Y must be L^*-S-connected space.

Proposition (3-10):- If f: (X, τ, T) → (Y, σ, L) be (ST, L)-S-continuous onto function and if X is ST-S-connected space, then Y is SL-S-connected space.

Proof:- Suppose Y is ST-S-disconnected space. Then, there exist two disjoint non-empty SL-S-open sets A, B in Y such that Y = A ∪ B, by definition (2-15) ⇒ (every SL-S-open is L-S-open) ⇒ A and B are also L-S-open sets in Y. Since f is (ST, L)-S-continuous onto function. Then, $f^{-1}(A)$, $f^{-1}(B)$ are two ST-S-open sets in X such that $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A) \cap f^{-1}(B) = \phi$

Thus, X is ST-S-disconnected space (which are contradiction)Since X is ST-S-connected space. Hence, Y must be SL-S-connected space.

Similarly, we prove the following proposition:

Proposition (3-11):- If f: (X, τ, T) → (Y, σ, L) be (ST, L)-S-continuous onto function and if X is ST-S-connected, then Y is L-S-connected space.

Corollary (3-12):- If f: (X, τ, T) → (Y, σ, L) is (T^*, L)-S-continuous onto function and if X is ST-S-connected, then Y is [L^*-S-connected space, SL-S-connected space, L-S-connected space] respectively.

Proof:-This follows from Proposition (3-2) (every (T*, L)-S-continuous is (ST, L)-S-continuous) ⇒ f is (ST, L)-S-continuous⇒ by using Propositions (3-9), (3-10) and (3-11) we get Y is (L^*-S-connected, SL-S-connected and L-S-connected) spaces respectively.

Proposition (3-13):- If f: (X, τ, T) → (Y, σ, L) be (T^*, L)-S-continuous onto function and if X is T^*-S-connected space, then Y is L^*-S-connected space.

Proof:-

Suppose Y is L^*-S-disconnected space. Then, there exist two disjoint non-empty L^*-S-open sets A and B in Y such that Y = A ∪ B. by definition (2-15) ⇒ (every L^*-S-open is L-S-open)⇒ A and B are L-S-open sets in Y

Since f is (T^*, L)-S-continuous onto function.Then, $f^{-1}(A)$, $f^{-1}(B)$ are two T^*-S-open sets in X such that $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

where $f^{-1}(A) \cap f^{-1}(B) = \phi$ .Thus, X is T^*-S-disconnected space (which are contradiction)Since X is T^*-S-connected space. Hence, Y must be L^*-S-connected space.

Proposition (3-14):- If f: (X, τ, T) → (Y, σ, L) be (T^*, L)-S-continuous onto function and if X is T^*-S-connected space, then Y is SL-S-connected space.

Proof:-

Suppose Y is SL-S-disconnected space .Then, there exist two disjoint non-empty SL-S-open sets A, B in Y such that $Y = A \cup B$,by definition (2-15) $\Rightarrow$ (every SL-S-open is L-S-open)$\Rightarrow$ A and B are L-S-open sets in Y. Since f is (T^*, L)-S-continuous onto function.$\Rightarrow$ $f^{-1}(A)$, $f^{-1}(B)$ are two T^*-S-open sets in X such that

$X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A) \cap f^{-1}(B) = \phi$

Thus, X is T^*-S-disconnected space (which are contradiction). Since X is T^*-S-connected space. Hence, Y must be SL-S-connected space.

Proposition (3-15):- If f: $(X, \tau, T) \rightarrow (Y, \sigma, L)$ be (T^*, L)-S-continuous onto function and if X is T^*-S-connected space, then Y is L-S-connected space.

Proof:-

Suppose Y is L-S-disconnected space. Then, there exist two disjoint non-empty L-S-open sets A, B in Y such that $Y = A \cup B$. Since f is (T^*, L)-S-continuous onto function.$\Rightarrow$ $f^{-1}(A)$, $f^{-1}(B)$ are two T^*-S-open sets in X such that

$X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A) \cap f^{-1}(B) = \phi$

Thus, X is T^*-S-disconnected space (which are contradiction)Since X is T^*-S-connected space. Hence, Y must be L-S-connected space.

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