حول التقارب –T للشبكة

On T-Convergence Of Net

أ.د. هادي جابر مصطفى
قسم الرياضيات / كلية التربية
الجامعة المستنصرية

د. حيدر جبر علي
قسم الرياضيات /كلية العلوم
الجامعة المستنصرية

المستخلص

في هذا البحث قمنا بطرح ودراسة بعض الخواص التبولوجية للتقارب –T بالنسبة للشبكة ، حيث ان T هو عبارة عن مؤثر مقترن بالتبولوجي T والمعرف على المجموعة غير الخالية X .

Abstract

In this paper we introduce and study properties of T-convergence of net , where T is an operator associated with the topology Γ on a nonempty set X**.**

Introduction

Let (X, Γ, T) be an operator topological space where X is a nonempty set and Γ is a topology on X and T is an operator associated with the topology with the topology Γ on X.

We introduce and study T-convergence of net in (X, Γ, T) ,we obtain the theory of θ–convergence of net in (X , Γ)[1] as a special case when Tis the usual closure operator.

1-Preliminaries

The symbols X and Y denote topological spaces with no separation axioms assumed unless explicitly stated.

Definition 1-1[2]:

Let (X, Γ) be a topological space a map T : Γ→ P(X) is called an operator associated with Γ if U ⊆ T(U) where U∈ Γ.

Definition 1-2[2]:

Let (X, Γ, T) , (Y, Γ^*, L) be two operator topological spaces. A map f: (X, Γ, T) → (Y, Γ^*, L) is said to be (T, L) continuous at x ∈ X if for each V∈ Γ^*, f(x) ∈ V, there exists U∈ Γ , x ∈ U such that f(T(U)) ⊆L(V). Notice that when T and L are the closure operators we get the definition [3] of θ −continuous function f:(X, Γ) → (Y, Γ^*).

2- T- convergence of net

Definition 2-1[4]:

A net in a set X is a function P: Λ → X, where Λ is some directed set. A subset of a net P: Λ → X , is the composition $P\circ\varphi$, where $\circ\varphi$: M → Λ is An increasing cofinal function from a directed set M into 6 .that is ,

a) $\varphi(\mu_1) \le \varphi(\mu_2)$ wherever $\mu_1 \le \mu_2$ (φ is increasing).

b) for each λ ∈ Λ , there is some μ ∈ M such that $\lambda \le \varphi(\mu)$, (φ is cofinal in Λ).

Definition 2-2:

Let (X, Γ, T) be operator topological space, a net $(x_\alpha)_{\alpha\in\Omega}$ in X is said to be is said to be T- converge to a point x ∈ X if and only if for every open set U containing x, ∃ $\alpha_0 \in \Omega$, ∋ $x_\alpha \in T(U)$, ∀ $\alpha \ge \alpha_0$ and denoted by $x_\alpha \xrightarrow{T}$ x and x is called T- limit point , also x ∈ X is called T- cluster point to a net $(x_\alpha)_{\alpha\in\Omega}$ in X if and only if ∀ U ∈ Γ with x ∈ U and ∀ $\alpha_0 \in \Omega$, ∃ $\alpha_0 \in \Omega$ with $\alpha \ge \alpha_0$ such that $x_\alpha \in T(U)$ and denoted by $x_\alpha \overset{T}{\propto} x$

Remark 2-3:

Let (X, Γ, T) be an operator topological space and $(x_\alpha)_{a\in\Omega}$ be a net in X, x ∈X then:

1. If $(x_\alpha)_{\alpha\in\Omega}$ converges to point x, then $(x_\alpha)_{\alpha\in\Omega}$ is T- converge to x.

2. If $(x_\alpha)_{\alpha\in\Omega}$ is T- converge to x, then x is T- cluster point to $(x_\alpha)_{\alpha\in\Omega}$. the converse of above (1 and 2) is not true in general . To show that we give the following examples:

Example 2-4:

i. If X = R the set of real numbers and Γ is the co-countable topology, every sequence $\{S_n\}$ in X with $S_n \neq S_m$ whenever $n \neq m$ is not convergent. It follows that every sequence $\{S_n\}$ in X and every $S_0 \in X$, U $(R-\{S_n\}) \cup \{S_0\}$ is open neighborhood of S_0 and U has at most finite terms of $\{S_n\}$ i.e., $\{S_n\}$ is not convergent . Now it is clear that $\{\frac{n}{n+1}\}$ is divergent but $T : \Gamma \rightarrow P(X)$ is defined by T(U) = R for each U ∈ Γ, then $\{\frac{n}{n+1}\}$ is T-convergent.

ii. Let (R, Γ_u, T) be an operator topological space where R is the set of all real numbers , Γ_u is the usual topology on R and T is closure operator. Now the net $(n+(-1)^n n)_{n\in N}$ has 0 as T- cluster point but 0 is not a T-limit point .

Theorem 2-5:

Let (X, Γ, T) be an operator topological space and let $(x_\alpha)_{\alpha\in\Omega}$ be a net in X and suppose x_α is T-converge to x , then every subnet of $(x_\alpha)_{\alpha\in\Omega}$ is T-converge to X.

Proof:

Let $(x_{\alpha_\beta})_{\beta\in D}$ a subnet of $(x_\alpha)_{\alpha\in\Omega}$, if U∈ Γ with x ∈ U, since $x_\alpha \xrightarrow{T} x$, then $\exists\ \alpha_0 \in \Omega$ such that $x_\alpha \in T(u)\ \forall\ \alpha \geq \alpha_0$ but $(x_{\alpha_\beta})_{\beta\in D}$ is a subnet of $(x_\alpha)_{\alpha\in\Omega}$ and $\alpha_0 \in \Omega$, $\exists\ \beta_0 \in D$ such that $\alpha_{\beta_\circ} \geq \alpha_0$, i.e. , $x_{\alpha_{\beta_\circ}} \in T(u)$,then $\exists\ \beta_0 \in D$ such that $\forall\ \beta \geq \beta_0$ and from definition of subnet $\alpha_\beta \geq \alpha_{\beta_\circ} \geq \alpha_0$ so $\alpha_\beta \geq \alpha_0$, then $x_{\alpha_\beta} \in T(u)$. Therefore , $(x_{\alpha_\beta})_{\beta\in D}$ is T-converge to *x*.

Theorem 2-6:

Let (X , Γ, T) be an operator topological space , let A ⊆ Y ,if A is T-open ,then no net in X \ T(A) can T-converge to a point in A .

Proof:

Let A be a T-open set in X and suppose that there is a net $(x_{\alpha})_{\alpha \in \Omega}$ in X \ T(A) which T-converge to x ∈A , since A is T-open we can choose U∈ Γ with x ∈U such that x ∈T(U) ⊆ A ⊆ T(A), i.e., ∃ $\alpha_0 \in \Omega$ such that , $x_{\alpha} \in$ T(A), but $(x_{\alpha}) \in$ X \ T(A) which is a contradiction

Corollary 2-7:

Let (X, Γ, T) be operator topological space and A ⊆ X , if A is T-closed then no net in X \ (T(X \A)) can T-converge to a point in X \ A.

Theorem 2-8:

Let (X, Γ, T) be operator topological space and A ⊆ X , if there is no net in X \ A can T-converge to a point in A, then A is T-open.

Proof:

suppose A is not T-open, then ∃ x ∈ A and ∀ U∈ Γ ∋ x ∈ U and T(U) ⊄ A then T(U) ∩ (X \ A) $\neq \phi$, pick $x_{T(u)}$ ∈T(U) ∩ (X \ A) , let $T(O_x)$ = { T(U) : U∈ Γ , x ∈U }, then $T(O_x)$ is directed by the inclusion relation ⊆ , then $(X_{T(u)})_{T(u) \in T(Ox)}$ is a net in X \ A and it is T-converge to x , but this is contradiction, so A is T-open.

Theorem 2-9:

Let f: (X, Γ, T) → (Y, Γ^*, L) be a function from an operator topological space (X, Γ, T) to operator topological space (Y, Γ^*, L) where T and L are monotone operators, let $(x_{\alpha})_{\alpha \in \Omega}$ be a net in X then is f is (T, L) - continuous at x∈X if and only if $f(x_{\alpha})$ is L- converge to f(x), whenever $(x_{\alpha})_{\alpha \in \Omega}$ is T-converge to x **.**

Proof:

Let f be a (T,L) – continuous at x and let $(x_{\alpha})_{\alpha \in \Omega}$ is T-converge to x, let V∈ Γ^* such that f(x) ∈V , since f is (T,L) – continuous, then ∃ U∈ Γ , x ∈U ∋ f(T(U)) ⊆ L(V) but $(x_{\alpha})_{\alpha \in \Omega}$ is T-converge to x, then ∃ $\alpha_0 \in \Omega$ ∋ x_{α} ∈T(U) ∀ $\alpha \geq \alpha_0$, then $f(x_{\alpha}) \in$ f(T(U)) ⊆ L(V) ∀ $\alpha \geq \alpha_0$, thus f (x_{α}) is L- converge to f(x) .Conversely , suppose f is not (T,L)- continuous at x, i.e. ∃ V∈ Γ^* , f(x) ∈V such that ∀ U∈ Γ , x ∈U, then f(T(U)) ⊄ L(V) so pick $x_{T(u)}$ ∈T(U) ∋ $f(X_{T(u)}) \notin$ L(V), then $(X_{T(u)})_{T(u) \in T(Ox)}$ is a net in X and $X_{T(u)}$ is T- converge to x and thus

$f(X_{T(u)})$ is is L- converge to f(x), then $f(x_{T(u)}) \in L(V)\ \forall\ T(U) \geq T(U_0)$ for some $T(U_0) \in T(O_x)$ but this is contradiction, then f is (T,L)- continuous at x.

Theorem 2-10:

Let (X, Γ, T) be an operator topological space and $A \subset X$. then $a \in X$ is a T-limit point of A if and only if and only if there exists a net $F:D \rightarrow X$ which is in A\ {a} and T- converge to a.

proof:

suppose ∃ net $F:D \rightarrow X$ which is A\ {a} and T- converge to a, if $U \in \Gamma_x$ with $a \in U$, then F is eventually in T(U) so that $T(U) \cap A \setminus \{a\} \neq \phi$. It follows that a is a T-limit point of A. Conversely ,suppose a is a T-limit point of A, then every open set U in X containing a ,T(U) containing at least one point $x \in A$ such that $x \neq a$, now let D = {(x, T(U)): U is an open set containing a and $x \in A \cap T(U)$, $x \neq a$ } we define $(x_1, T(U_1)) \geq (x_2,T(U_2))$ if and only if $T(U_1) \subseteq T(U_2)$. It is clear that (D, ≥) is a directed set , we may define a net $F : D \rightarrow X$ as F(x , T(U)) = x , then by its definition the net F is in A\ {a}. we claim that F is T- converge to a , thus if $U \in \Gamma$ with $a \in U$, then $\exists\ x \in A \setminus \{a\} \cap T(U_0)$. put $\alpha_o = (x_0 , T(U_0)) \in D$, Now $\forall\ \alpha=(x, T(U)) \geq \alpha_o = (x_0, T(U))$,$T(U) \subseteq T(U_0)$ so that $F(\alpha) = x \in T(U) \subseteq T(U_0)$, therefor , F is T-converge to a.

Theorem 2-11:

Let (X, Γ, T) be an operator topological space and let $\{X\alpha\}_{\alpha \in D}$ be a net in X, $a \in X$, then $x_\alpha \overset{T}{\propto} a$ if and only if ∃ a subnet$\{x_{\alpha\beta}\}_{\beta \in D}$ of $\{X\alpha\}_{\alpha \in D}$ which T- converge to a .

Proof:

Assume that $a \in X$ is a T-cluster point of a net$\{X\alpha\}_{\alpha \in D}$. To show that ∃ subnet of $\{X\alpha\}_{\alpha \in D}$. which T-converge to a , put $D_1 = \{(\alpha, T(U)):$ where $U \in \Gamma$ with $a \in U$ and $x_\alpha \in T(U))\}$ .Cleary that $D_1 \neq \phi$ (since $x_\alpha \overset{T}{\propto} a$) , now ordered D_1 by the relation ≥ which is defined by $(\alpha_1 , T(U_1)) \geq (\alpha_2 , T(U_2))$ iff $\alpha_1 \geq \alpha_2$ and $T(U_1) \subseteq T(U_2)$. It is easy that $(D_1 , \geq)$ is directed set and the map $x: D_1 \rightarrow X$ such that $x(\alpha , T(U)) = x_\alpha$ is subnet of $\{x_\alpha\}_{\alpha \in D}$. To prove that $\{x(\alpha , T(U))\}_{(\alpha, T(U)) \in D1}$ is T-converges to a . If $U_0 \in \Gamma$ with

$\alpha \in U_0$, since $x_\alpha \overset{T}{\propto}$ a , then $\forall\ \alpha \in D$, $\exists\ \alpha_0 \in D$ with $\alpha_0 \geq \alpha$ such that $X_{\alpha 0} \in T(U_0)$.

Put $\beta_0 = (\alpha , T(U_0)) \in D_1$ so $\forall\ \beta = (\alpha , T(U)) \geq \beta_0 = (\alpha_0 , T(U_0))$ if and only if $T(U) \subseteq T(U_0)$, then $x_{\beta=(\alpha , T(U))} = x_\alpha \in T(U) \subseteq T(U_0)$.

Therefore, $\{x(\alpha , T(U))\}_{(\alpha , T(U)) \in D1}$ is T-converges to a .Conversely, suppose that $\{x_{\alpha\beta}\}_{\beta \in D1}$ is a subnet of a net $\{x_\alpha\}_{\alpha \in D}$ and $x_{\alpha\beta} \xrightarrow{T} \alpha \in X$, to show that $x_\alpha \overset{T}{\propto}$ a

Let $U \in \Gamma$ with $a \in U$, $\alpha_0 \in D$ by definition of subnet $\exists\ \beta_0 \in D_1$ such that $\alpha_\beta \geq \alpha_0$, but $x_{\alpha\beta} \xrightarrow{T} \alpha \in T(U)$, then $\exists\ \beta_1 \in D_1$ such that $x_{\alpha_\beta} \in T(U)\ \forall\ \beta \geq \beta_1$, since $\beta_0, \beta_1 \in D_1$, then $\exists\ \beta_2 \in D_1$ such that $\beta_2 \geq \beta_0$ (i.e. $\alpha_{\beta_2} \geq \alpha_{\beta_0} \geq \alpha$) and $\beta_2 \geq \beta_1$ therefore, a is T-cluster point of $\{x_\alpha\}_{\alpha \in D}$.

Definition 2-12[2]:

Let (X, Γ, T) be an operator topological space , we say that (X, Γ) is a T- Hausdorff space if for every pair x ,y of distinct points of X there exist two open sets U,V such that $x \in U$, $y \in V$ and $T(U) \cap T(V) = \phi$.

Clearly if the space (X, Γ) is T- Hausdorff, it is also Hausdorff, but the converse is not true in general for example : if X=R , the real line with usual topology then $(X, \Gamma)$is Hausdorff and if $T: \Gamma \rightarrow P(X)$ which is defined by $T(U) = R$ for each $U \in \Gamma$, then (X, Γ, T) is not T- Hausdorff .

Theorem 2-13:

Let (X, Γ, T) be an operator topological space , X is T- Hausdorff space if and only if every T-convergent net in X has unique T-limit point.

Proof:

Given X is T- Hausdorff and suppose $\exists$ a net $\{X\alpha\}_{\alpha \in D}$ in X such that $x_\alpha \xrightarrow{T} x_0$ and $x_\alpha \xrightarrow{T} y_0$, $x_0 \neq y_0$, thus $\exists U, V \in \Gamma$ with $x_0 \in U$ and $y_0 \in V$ such that $T(U) \cap T(V) = \phi$. Since $x_\alpha \xrightarrow{T} x_0 \in U$, thus $\exists\ \alpha_1 \in D$

such that $x_\alpha \in T(U)\ \forall \alpha \geq \alpha_1$, also $x_\alpha \xrightarrow{T} y_0 \in V$, then $\exists\ \alpha_2 \in D$ such that $x_\alpha \in T(V)\ \ \forall \alpha \geq \alpha_2$, but $\alpha_1, \alpha_2 \in D$, then $\exists\ \alpha_3 \in D$ such that $\alpha_3 \geq \alpha_1$ and $\alpha_3 \geq \alpha_2$ so $\forall \alpha \geq \alpha_3$, $x_\alpha \in T(U)$and $x_\alpha \in T(V)$,then $x_\alpha \in T(U) \cap T(V)$, $\forall \alpha \geq \alpha_3$ i.e.,

$T(U) \cap T(V) \neq \phi$.which is a contradiction . Therefore, $x_\alpha \xrightarrow{T} x_0 = y_0$.

Conversely , suppose X is not T- Hausdorff, then $\exists\ x_0, y_0 \in X$ and $x_0 \neq y_0$, such that for each $U, V \in \Gamma$ with $x_0 \in U$ and $y_0 \in V$ and $T(U) \cap T(V) \neq \phi$.

Let N_{x0} , N_{y0} = { (T(N), (T(M)): N,M are neighborhoods of x_0 and y_0 respectively }.It is clear that (N_{x_0} , $\subseteq$) and (N_{y_0}, $\subseteq$) are directed sets ,hence for each $N \in N_{x_0}$ and $N^* \in N_{y_0}$, $N \cap N^* \neq \phi$ we define the directed set (N_{x_0} x N_{y_0}, $\succ\!\succ$) where $\succ\!\succ$ is a relation on N_{x_0} x N_{y_0} defined by $(N_1$ x $N^*_1)$, $\succ\!\succ$ $(N_2, N^*_2) \leftrightarrow N_1 \subseteq N_2$ and $N^*_1 \subseteq N^*_2$ also define the net x:(N_{x_0} x N_{y_0}, $\succ\!\succ$) $\rightarrow$X by $x((N, N^*)) \in (N \cap N^*) \neq \phi$.

Now if U is an open subset of X with $x_0 \in T(U)$, thus $T(U) \in N_{x_0}$ and we take a fixed $N^* \in N_{y_0}$, and hence $\alpha_0 = (T(U), N^*) \in N_{x_0}$ x N_{y_0}, also since $x(\alpha_0) = x\ T(U), N^*) \in (T(U) \cap N^*) \subseteq T(U)$, we have that for each $\alpha = (N_1, N^*_1) \succ\!\succ ((T(N), N^*) = \alpha_0 \leftrightarrow N_1 \subseteq T(U)$ and $N_1^* \subseteq N^*$ thus $x(\alpha) = x(N, N^*_1) \in (N_1 \cap N^*_1) \subseteq (T(U) \cap N^*) \subseteq T(U)$, so $x_\alpha \in T(U)$, $\forall \alpha \succ\!\succ \alpha_0$,then $x_\alpha \xrightarrow{T} x_0$ By the same way we show that $x_\alpha \xrightarrow{T} y_0$ which is a contradiction . therefore, X is T- Hausdorff

References

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