On order and type of entire functions represented by special monogenic functions

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Abstract

In the present paper we study the generalized order and generalized type of special monogenic functions having slow growth. The studied characterizations of generalized order and type of special monogenic functions have been obtained in terms of their Taylor's series coefficients.

Keywords: Clifford algebra, Clifford analysis, special monogenic functions, generalized order and type.

1- Introduction

Firstly, following Constales, Almeida and Krausshar (see [1] and [2]), we give some definitions and associated properties .Let m= $(m_1,m_2,\ldots,m_n) \in N_0^n$ be the n-dimensional multi-index and $x \in R^n$ then we define
$x^m=x_1^{m1}\ldots x_n^{mn}$, $m! =m_1! \ldots,m_n!$, $|m| = m_1+\ldots+m_n$ … (1)

By $\{e_1, e_2,\ldots,e_n\}$ we denote the canonical basis of the Euclidean vector space R^n. The associated real Clifford algebra CI_n is free algebra generated by R^n modulo $x^2=-\|x\|^2 e_0$, where e_0 is the neutral element with respect to multiplication of the Clifford algebra CI_n .In the Clifford algebra CI_n following multiplication rule holds :
$e_ie_j+e_je_i=-2\,\delta_{i,j}$, i,j =1,2,…,n. Where δ_{ij} is kronecker symbol. … (2)

A basis for Clifford algebra CI_n is given by the set $\{e_A : A\subseteq \{1,2,\ldots,n)\}$, with $e_A=e_{l_1}e_{l_2}\ldots,e_{l_r}$.

Where $1\le l_1\le l_2\ldots\le l_r\le n$,$e_\phi=e_0=1$.Each $a \in CI_n$ can be written in the form $a=\sum_{A\subseteq(1,2,..,n)} a_A\, e_A$,

With $a_A \in R$.The conjugation in Clifford algebra CI_n is defined by $\bar{a}=\sum_{A\subseteq(1,2,..,n)} a_A\, \bar{e}_A$ Where $\bar{e}_A=\bar{e}_{l_r}\,\bar{e}_{l_{r-1}}\ldots.\bar{e}_{l_1}$, and $\overline{e_j}=-\,e_j$ for j=1,

$2\ldots,n$, $\bar{e}_0=e_0=1$. The linear subspace span $_R\{1,e_1,e_2,\ldots,e_n\} \subseteq CI_n$ is the so called space of Para vectors $z = x_0 + x_1 e_1 + x_2 e_2 + \ldots\ldots + x_n e_n$ which we simply identify with R^{n+1}: Here $x_0=Sc(z)$ is scalar part and $x=x_1e_1+x_2e_2+\ldots+x_ne_n=Vec(z)$ is vector part of Paravector z: The Clifford norm of an arbitrary $a= \sum_{A\subseteq(1,2,\ldots,n)} a_A e_A$ is given by $\|a\| = (\sum_{A\subseteq(1,2,\ldots,n)} |a_A|^2)^{1/2}$. The generalized Cauchy–Riemann operator in R^{n+1} is given by $D= \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$. If $U\subseteq R^{n+1}$ is an open set then the function $g: U\rightarrow CI_n$ is called left (right) monogenic at a point $z \in U$ if $Dg(z) =0$ ($gD(z) = 0$). The functions which are left (right) monogenic in the whole space called left (right) entire monogenic functions.

Following Abul-Ez and Constales [4], we consider the class of monogenic polynomials p_m of degree $|\mathbf{m}|$, defined as

$$P_m(z)= \sum_{i+j=|m|}^{\infty} \frac{((n-1/2))i}{i!}\frac{((n+1)/2))j}{j!} (\bar{z})^i (z)^j \qquad \ldots (3)$$

Let ω_n be n-dimensional surface area of n +1-dimensional unit ball and let S^n be n- dimensional sphere. Then, the class of monogenic polynomials described in (3) satisfies (see [5], p. 1259)

$$\frac{1}{\omega_m} \int_s^n \overline{p_m}\, p_l(z)\, dS_z= k_m\, \delta_{|m||l|} \qquad \ldots (4)$$

Also following Abul-Ez and De Almeida [5], we have

$$\max_{\|z\|=r} \|P_m(z)\| =k_m r^m \qquad \ldots (5)$$

2. Preliminaries

In this section we give some definition which will be used in the next section

Definition (2.1),[5]: Let **Ω** be a connected open subset of R^{n+1}containing the origin and let (z) be monogenic in **Ω**. Then (z) is called special monogenic in **Ω**, if and only if its Taylor's series near zero has the form

$$g(z)=\sum_{|m|=0}^{\infty} p_m(z)c_m \quad , \quad c_m\in CI_n \qquad \ldots (6)$$

Definition(2.2),[5] : Let $g(z)=\sum_{|m|=0}^{\infty} p_m(z)c_m$ be a special monogenic function defined on a neighborhood of the closed ball $B(0,r)$. Then,

$$\|c_m\| \leq\frac{1}{\sqrt{k_m}} M(r,g)\, r^{-m} \qquad \ldots (7)$$

Where $M(r,g)=\max_{\|z\|=r}\|g(z)\|$ is the maximum modulus of g(z).

Definition(2.3),[5]: Let $g:R^{n+1} \to CI_{0n}$ be a special monogenic function whose Taylor's series representation is given by (6). Then, for $r > 0$ the maximum term of this special monogenic function is given by

$$\mu(r) = \mu(r,g) = \max_{|m|\geq 0}\{\|a_m\| k_m r^m\} \qquad \dots (8)$$

Also the index **m** with maximal length |**m**| for which maximum term is achieved is called the central index and is denoted by:

$$v(r) = v(r, f) = \mathbf{m}. \qquad \dots (9)$$

Definition (2.4),[5]: let $g:R^{n+1} \to CI_n$ be a special monogenic function whose Taylor's series representation is given by (6) Then, the order ρ and lower order λ of (z) are defined as :

$$\rho = \lim_{r\to\infty} \sup \frac{\log\log M(r,g)}{\log r} \qquad \dots (10)$$

$$\lambda = \lim_{r\to\infty} \inf \frac{\log\log M(r,g)}{\log r}$$

Definition (2.5),[5]: Let:$R^{n+1} \to CI_{0n}$ be a special monogenic function whose Taylor's series representation is given by (6) Then, the type σ and lower type of g are defined as :

$$\sigma = \lim_{r\to\infty} \sup \frac{\log M(r,g)}{r^{\rho}} \qquad \dots (11)$$

$$\omega = \lim_{r\to\infty} \inf \frac{\log M(r,g)}{r^{\rho}}$$

The concept of generalized order and generalized type for entire transcendental functions was given by Seremeta [6], Kapoor and Nautiyal [3], Hence, let L^0 denote the class of functions h(x) satisfying the following conditions:

(i) h(x) is defined on [a;∞) and is positive, strictly increasing, differentiable and tends to ∞ as x → ∞;

(ii) $\lim_{x\to\infty} \frac{h[\{1+\frac{1}{\Psi(x)}\}x]}{h(x)} = 1$, for every function Ψ (x) such that Ψ(x) →∞ as x → ∞.The functions of the form f(x) = ax + b; 0 < a < ∞; 0 < b < ∞ are in class L^0. Let ʌ denote the class of functions h(x) satisfying conditions (i) and

(iii) $\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$

For every c>0, that is h(x) is slowly increasing .The functions of the form f(x)=log(ax), $0<a<\infty$,are in class Λ . Let Ω be the class of function h(x) satisfying condition (i) and

(iv) there exist a function $\delta(x) \in \Lambda$ and constants x_0 , K_1 and K_2 such that

$$0 \le k_1 \le \frac{d\{h(x)\}}{d\{\delta(\log x)\}} \le k_2 < \infty,$$

For all $x>x_0$, The functions of the form $f(x) = \delta(\log x)$, $\delta \in \Lambda$ are in class Λ. (see [3]). Let $\overline{\Omega}$ be the class of functions h(x) satisfying (i) and

(v) $\lim_{x\to\infty} \frac{d\{h(x)\}}{d\{\log x\}} = K$, $0<K<\infty$. The functions of the form $f(x) = \log x + a (\log\log x)^b$; $0 < a < \infty$; $0<b<\infty$ are in class $\overline{\Omega}$. (See [3]) .

Kapoor and Nautiyal [3] showed that classes Ω and $\overline{\Omega}$ are contained in Λ and $\Omega \cap \overline{\Omega} = \phi$. For an entire monogenic function g(z) and functions $\alpha(x)$ either belongs to Ω or $\overline{\Omega}$, we define the generalized order $\rho(\alpha, g)$ of g(z) and generalized type $\sigma=(\alpha,\rho,g)$ as :

$$\rho(\alpha,g)=\lim_{r\to\infty}\sup\frac{\alpha[\log M(r,g)]}{\alpha(\log r)} \quad \ldots (12)$$

$$\sigma=(\alpha,\rho,g)=\lim_{r\to\infty}\sup\frac{\alpha[\log M(r,g)]}{[\alpha(\log r)]^{\rho}}$$

3. Main Results

Now we prove

Theorem (3.1):

Let $g:R^{n+1}\to Cl_{0n}$ be a special monogenic function whose Taylor's series representation is given by (6) if $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$, then the generalized order $\rho(\alpha,g)$,$(1<\rho(\alpha,g)<\infty)$ of g(z) is given as : -

$$\rho(\alpha,g)-1=\lim_{|m|\to\infty}\sup\frac{\alpha(|m|)}{\alpha\{\log\|c_m\|^{\frac{-1}{|m|}}\}} \quad \ldots (13)$$

Proof: -

Write

$$\Theta=\lim_{|m|\to\infty}\sup\frac{\alpha(|m|)}{\alpha\{\log\|m\|^{\frac{-1}{|m|}}\}} \quad \ldots (14)$$

Now first we prove that $\rho-1 \ge \Theta$. The coefficients of a monogenic Taylor's series satisfy Cauchy's inequality, that is

$$\|c_m\| \leq \frac{1}{\sqrt{k_m}} M(r,g) \quad r^{-|m|} \qquad \ldots (15)$$

Also from (12) ,for arbitrary $\mathcal{E}>0$ and all $r>r_0(\mathcal{E})$, we have

$$M(r,g) \leq \exp[\alpha^{-1}\{\acute{p}\,\alpha(\log r)\}] \qquad \ldots (16)$$

Where $\rho'=\rho+\mathcal{E}$

Now from inequality (15) we have

$$\|c_m\| \leq \frac{1}{\sqrt{k_m}} \quad r^{-|m|} \exp[\alpha^{-1}\{\acute{p}\,\alpha(\log r)\}]$$

Since $(\frac{1}{\sqrt{k_m}}) \leq 1$ (see [5], pp.148) so the above inequality reduces to ….. (17)

$$\|c_m\| \leq r^{-|m|} \exp[\alpha^{-1}\{\acute{p}\,\alpha(\log r)\}]$$

Or

$$\|c_m\| \leq \exp[-|m|\log r + \alpha^{-1}\{\acute{p}\,\alpha(\log r)\}] \qquad \ldots (18)$$

Since $\alpha(x)$ is an increasing function of x, we define $r=r(|m|)$ as the unique root of equation

$$\alpha\left[\frac{|m\ |\log r}{\acute{p}}\right] = \acute{p}\,\alpha(\log r) \qquad \ldots (19)$$

For large values of $|m|$, we have

$$\alpha(c|m|) \approx \alpha(|m|)$$

$$\rightarrow \alpha(c|m|) \approx \alpha(|m|)\{1+o(1)\}$$

$$\rightarrow \alpha(c|m|) \approx \alpha(|m|)\{1+\frac{\alpha(c)}{\alpha(|m)|}\}$$

$$\rightarrow \alpha(c|m|) \approx \alpha(|m|) + \alpha(c)$$

Thus for large values of $|m|$ from equation (19) we have

$$\acute{p}\alpha(\log r) \approx \alpha(|m|) + \alpha(\log r) - \alpha(\acute{p})$$

Or $$\alpha(\log r) \approx \frac{\alpha(|m)|}{\acute{p}-1}\ \{\frac{1-\alpha(\acute{p})}{\alpha(|m|)}\}$$

Or $$\alpha(\log r) \approx \frac{\alpha(|m)|}{\acute{p}-1}\ \{1+o(1)\}$$

Or $$\log r \approx \alpha^{-1}\{\frac{1}{\acute{p}-1}\alpha(|m|)\} = F(|m|\ \frac{1}{\acute{p}-1}) \qquad \ldots (20)$$

using (19)and (20) in (18) we get

$$\|c_m\| \leq \exp\,[-|m|F+(\frac{|m|}{\acute{p}})F]$$

Or $$\frac{\acute{p}}{\acute{p}-1}\log\{\|c_m\|^{\frac{-1}{|m|}}\} \geq \alpha^{-1}\{\frac{1}{\acute{p}-1}\alpha(|m|)\}$$

Or $$\frac{\alpha(|m|)}{\alpha\{\frac{\acute{p}}{\acute{p}-1}\log\{\|c_m\|\}^{\frac{-1}{|m|}}\}} \leq \acute{p}-1$$

Or $$\frac{\alpha(|m|)}{\alpha[\log\{\|c_m\|^{\frac{-1}{|m|}}\}]} \leq (\acute{p}-1) \times \frac{\alpha[\frac{\acute{p}}{\acute{p}-1}\log\{\|c_m\|^{\frac{-1}{|m|}}\}]}{\alpha[\log\{\|c_m\|^{\frac{-1}{|m|}}\}]}$$

Since $\alpha(cx) \approx \alpha(x)$ as $x \to \infty$, proceeding to limit as $|m| \to \infty$ we get

$$\Theta \leq \acute{p}-1$$

Since $\mathcal{E} > 0$ is arbitrarily small we finally get

$$\Theta \leq \rho-1 \qquad \dots (21)$$

Now, we will prove that $\Theta \geq \rho$. If $\Theta=\infty$, then there is nothing to prove. So let us assume that $0 \leq \Theta \leq \infty$, Therefore, for a given $\mathcal{E}>0$ there exist $n_0 \in N$ such that for all multi –indices m with $|m|>n_0$,we have

$$\frac{\alpha(|m|)}{\alpha[\log\{\|c_m\|^{\frac{-1}{|m|}}\}]} \leq \Theta+\mathcal{E}=\tilde{O}$$

Or

$$\|c_m\| \leq \exp\,[-|m|\,\alpha^{-1}\{\alpha(|m|)/\tilde{O}\}]$$

Now from the property of maximum modulus, we have

$$M(r,g) \leq \sum_{|m|=0}^{\infty} \|c_m\|\; k_m\, r^{|m|}$$

Or $$M(r,g) \leq \sum_{|m|=0}^{\infty} k_m\; r^{|m|} \exp[-|m|\,\alpha^{-1}\{\frac{|m|}{\tilde{O}}\}]$$

Now for $r = \max\{1,\exp(\alpha^{-1}(\frac{\alpha(n_0+1)}{\tilde{O}})/(n+1)\}$, we have

$$M(r,g) \leq A_1\, r^{n0} + \sum_{|m|=n_{0+1}}^{\infty} k_m\; r^{|m|} \exp\,[-|m|\,\alpha^{-1}\{\alpha(|m|)/\tilde{O}\}] \quad \dots (22)$$

Where A_1 is positive real constant.

We take $N(r)=[\alpha^{-1}\{\tilde{O}\,\alpha[\log\{(n+1)r\}]\}]$ where [x] denotes the integer part of $x \geq 0$.

Since $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$,the integer N(r) is well defined. Now if r is sufficient large then from (22) we have

$$M(r,g)\leq A_1 r^{n0}+r^{N(r)}\times \sum_{n0+1<|m|<N(r)}^{\infty} k_m \exp[-|m|\alpha^{-1}\{\alpha(|m|/\tilde{O}\}]$$
$$+\sum_{|m|>N(r)} k_m \exp[-|m|\ \alpha^{-1}\{\alpha(\frac{|m|}{\tilde{O}})\}] \quad \text{....(23)}$$

Now the first series in (22) can rewritten as

$$\sum_{p=1}^{\infty}(\sum_{|m|=p}^{\infty} k_m) \exp[-p\alpha^{-1}\{\alpha(p)/\tilde{O}] \quad \text{...(24)}$$

Now from ([2], lemma 1), we have

$$\lim_{p\to\infty} \sup(\sum k_m)^{\frac{1}{p}} = n$$

Hence we have

$$\lim_{p\to\infty} \sup[(\sum_{|m|=p} k_m) \exp[-p\alpha^{-1}\{\alpha(p)/\tilde{O}\}]^{1/p} = n \lim_{p\to\infty} \sup \exp[-\alpha^{-1}\{\alpha((p)/\tilde{O}\}]=0$$

Hence the series (24) converges to a positive real constant A_2.so from (23) we get

$$M(r,g)\leq A_1 r^{n0}+A_2 r^{N(r)}+ \sum_{|m|>N(r)} k_m r^{|m|} \exp[-|m|\ \alpha^{-1}\{\alpha(|m|/\tilde{O}\}]$$

Or $\quad M(r,g)\leq A_1 r^{n0}+A_2 r^{N(r)} \sum_{|m|>N(r)} k_m r^{|m|} \exp[-|m| \log\{(n+1)r\}]$

Or $M(r,g)\leq A_1 r^{n0}+A_2 r^{N(r)} + \sum_{|m|>N(r)} k_m (\frac{1}{n+1})^{|m|}$ . ..(25)

The series in (25) can rewritten as

$$\sum_{p=1}^{\infty}(\sum_{|m|=p}^{\infty} k_m) (\frac{1}{n+1})^{p} \quad \text{... (26)}$$

So we have

$$\lim_{p\to\infty} \sup [(\sum_{|m|=p} k_m) (\frac{1}{n+1})^{p}]^{1/p} = \frac{n}{n+1} <1$$

Hence the series (26) converges to a positive real constant A_3.So from (25) we get

$$M(r,g)\leq A_1 r^{n}0+A_2 r^{N(r)}+A_3$$

Since $N(r)\to\infty$ as $r\to\infty$ so we can write above inequality as

$$\text{Log } M(r,g) \leq [1+o(1)]N(r)\log r$$

Or $\quad \text{Log } M(r,g)\leq[1+o(1)][\alpha^{-1}\{\tilde{O}\alpha[\log\{(n+1)r\}]\}]\log r$

$\leq [1+o(1)][\alpha^{-1}\{\tilde{O}\,\alpha[\log\{(n+1)r\}]\}]\times[\alpha^{-1}\{\alpha(\log\{(n+1)r\}]]$

$\leq [1+o(1)][\alpha^{-1}\{(\tilde{O}+1)\alpha[\log\{(n+1)r\}]\}]$

Or $\alpha[\log M(r,g)]\leq(\tilde{O}+1)\alpha[\log(n+1)r\}$

Or $\frac{\alpha[\log M(r,g)]}{\alpha(\log r)} \leq (\tilde{O}+1)\frac{[\alpha[\{1+o(1)\}\log r]}{\alpha(\log r)}$

Proceeding to limits as $r\rightarrow\infty$ and using properties of $\alpha(x)$,we get

$$\rho\leq\tilde{O}+1$$

Since $\mathcal{E}>0$ is arbitrarily small, we finally get

$$P-1\leq\Theta \qquad \ldots(27)$$

Combining (21) and (27) ,we get (13).hence theorem 1 is proved . Next ,we prove the following

Theorem (3.2):

Let $g:R^{n+1}\rightarrow CI_{0n}$ be a special monogenic function whose Taylor's representation is given by (6) Also if $\alpha(x)$ either belongs to **Ω** or to $\overline{\Omega}$ and $o<\rho<\infty$,then the generalized type $\sigma=(\alpha,\rho,g)$ of g(z) is given by

$$\sigma(\alpha,\rho,g)-1=\lim_{|m|\rightarrow\infty} \sup \frac{\alpha(\frac{|m|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\log(\|c_m\|)^{\frac{-1}{|m|}}\}]^{\rho-1}} \qquad \ldots(28)$$

proof:

write $\sigma=\sigma(\alpha,\rho,g)$ and

$$\eta=\lim_{|m|\rightarrow\infty} \sup \frac{\alpha(\frac{|m|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\log(\|c_m\|)^{\frac{-1}{|m|}}\}]^{\rho-1}} \qquad \ldots(29)$$

First we prove that $\eta\leq\sigma-1$, the coefficients of special monogenic function satisfy Cauchy's inequality,
that is

$$\|c_m\| \leq\frac{1}{\sqrt{k_m}} M(r,g)\; r^{-|m|} \qquad \ldots(30)$$

Also from (12) ,for arbitrary $\mathcal{E}>0$ and all $r>r_0(\mathcal{E})$ we have

$$M(r,g) \leq\exp([\alpha^{-1}[\sigma'(\alpha(\log r)\}^{\rho}) \qquad \ldots(31)$$

Where $\sigma'=\sigma+\mathcal{E}$

Since $(\frac{1}{\sqrt{k_m}}) \leq 1$ (see [5],pp.148) …(32)

So from eq(31)and(32) then eq(30)reduces to

$$\|c_m\| \leq r^{-|m|} \exp(\alpha^{-1}[\sigma'\{\alpha(\log r)\}^{\rho}])$$

Or $$\|c_m\| \leq \exp(-|m| \log r + \alpha^{-1}[\sigma'\{\alpha(\log r)\}^{\rho}]) \quad \ldots(33)$$

Let r=r(|m|) be unique root of the equation

$$\alpha\left[\frac{|m|\log r}{\rho}\right] = (\sigma'-1)\{\alpha(\log r)\}^{\rho} \quad \ldots(34)$$

Then for all large values of |m|, we have

$$\log r \approx \alpha^{-1}[\{ \frac{1}{\sigma'-1} \alpha(|m|)/\rho \}^{1/\rho-1}]=G(|m| /\rho , \frac{1}{\sigma'-1} , \rho-1) \quad \ldots (35)$$

Using (34) and (35) in (33) we get

$$\|c_m\| \leq \exp[-|m|G+(\frac{|m|}{\rho})G]$$

Or $$\frac{\rho}{\rho-1} \log \{\|c_m\|^{\frac{-1}{|m|}} \} \geq \alpha^{-1}[\{\frac{1}{\sigma'-1} \alpha(|m|/\rho)\}^{1/\rho-1}]$$

Or $$\frac{\alpha(|m|/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\log(\|c_m\|^{\frac{-1}{|m|}})\}]^{\rho-1}} \leq \sigma'-1$$

Proceeding to limit as $|m| \to\infty$, we get

$$\eta =\lim_{|m|\to\infty} \sup \frac{\alpha(\frac{|m|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\log(\|c_m\|)^{\frac{-1}{|m|}}\}]^{\rho-1}} \leq \sigma'-1$$

since $\varepsilon>0$ is arbitrarily small we finally get

$$\eta\leq(\sigma-1) \quad \ldots (36)$$

Now, we will proved that $\sigma-1\leq\eta$. If $\eta=\infty$, then there is nothing to prove. So let us assume that $0\leq\eta\leq\infty$, therefore, for all $\varepsilon >0$ there exist $n_0 \in N$ such that for all multi –indices m with $|m|>n_0$,we have

$$0 \leq \frac{\alpha(\frac{|m|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\log(\|c_m\|)^{\frac{-1}{|m|}}\}]^{\rho-1}} \leq \eta+\varepsilon=\eta' \quad \ldots(37)$$

Or

$$\|c_m\| \le k_m \exp(-\frac{\rho-1}{\rho}|m|\,\alpha^{-1}[\{\frac{1}{\eta'}\alpha(|m|/\rho)\}^{1/\rho-1}])$$

Now from the property of maximum modulus, we have

$$M(r,g) \le \sum_{|m|=0}^{\infty} \|c_m\|\, r^{|m|}$$

Or $M(r,g) \le \sum_{|m|=0}^{\infty} \|c_m\| r^{|m|} + \sum_{|m|=n_{0+1}} k_m\, r^{|m|} \exp(-\frac{\rho-1}{\eta'}|m|\,\alpha^{-1}[\{\frac{1}{\eta'}\alpha(\frac{|m|}{\rho})\}^{1/\rho-1}])$

Now for r>1,we have $M(r,g) \le B_1 r^{n0} + \sum_{|m|=n_{0+1}} k_m\, r^{|m|} \exp(-\frac{\rho-1}{\rho}|m|\,\alpha^{-1}[\{\frac{1}{\eta'}\alpha(\frac{|m|}{\rho})^{1/\rho-1}])$ …(38)

Where B_1 is a positive real constant .we take

$$N(r)=[\rho\,\alpha^{-1}\{\eta'(\alpha[\frac{\rho}{\rho-1}\log\{(n+1)r\}])^{\rho}-1]$$

Where [x] denotes the integer part of x≥0. Since α(x) either belongs to Ω or to $\overline{\Omega}$ the integer N(r) is well defined. Now if r is sufficiently large, then from (38) we have $M(r,g) \le B_1 r^{n0} + r^{N(r)} \times \sum_{n_{0+1<N(r)}} k_m \exp(-\frac{\rho-1}{\rho}|m|\,\alpha^{-1}[\{\frac{1}{\eta'}\alpha(\frac{|m|}{\rho})\}^{1/\rho-1}]] + \sum_{|m|>N(r)} k_m\, r^{|m|} \exp(-\frac{\rho-1}{\rho}|m|\alpha^{-1}[\{\frac{1}{\eta'}\alpha(\frac{|m|}{\rho})\}^{1/\rho-1}\})$ … (39)

Now the first series in (39) can be rewritten as:

$$\sum_{p=1}^{\infty}(\sum_{|m|=p}^{\infty} k_m)\exp(-\frac{\rho-1}{\rho}p\alpha^{-1}[\{\frac{1}{\eta'}\alpha(p/\rho)\}])^{1/\rho-1} \quad \text{…(40)}$$

Now from ([2], Lemma 1), we have

$$\lim_{p\to\infty}\sup(\sum_{|m|=p} k_m)^{1/p}=n$$

Hence we have

$$\lim_{p\to\infty}\sup[(\sum_{|m|=p} k_m)\exp(-\frac{\rho-1}{\rho}\rho\alpha^{-1}[\{\frac{1}{\eta'}\alpha(p/\rho)\}^{1/\rho-1}]]^{1/\rho}$$

$$=n\lim_{p\to\infty}\sup\exp(-\frac{\rho-1}{\rho}p\alpha^{-1}[\{\frac{1}{\eta'}\alpha(p/\rho)\}^{1/\rho-1}])=0$$

Hence the series (40) converges to a positive real constant B_2 .So from(39) we get

$$M(r,g) \le B_1 r^{n0} + B_2 r^{N(r)} + \sum_{|m|>N(r)} k_m\, r^{|m|}\exp(-\frac{\rho-1}{\rho}|m|\alpha^{-1}[\{\frac{1}{\eta'}\frac{\alpha(|m|)}{\rho}\}^{1/\rho-1}])$$

Or

$$M(r,g)\le B_1 r^{n0}+ B_2 r^{N(r)} + \sum_{|m|>N(r)} k_m \; r^{|m|}\exp[-|m| \log\{(n+1)r\}]$$

Or $$M(r,g)\le B_1 r^{n0}+ B_2 r^{N(r)} + \sum_{|m|=1} k_m \left(\frac{1}{n+1}\right)^{|m|} \qquad \ldots(41)$$

The series in (41) can we rewritten as

$$\sum_{p=1}^{\infty}\left(\sum_{|m|=p} k_m\right)\left(\frac{1}{n+1}\right)^{p} \qquad \ldots (42)$$

So we have

$$\lim_{p\to\infty} \sup \left[\sum_{|m|=p}^{\infty} (k_m) \left(\frac{1}{n+1}\right)^{\rho}\right]^{1/p} = \frac{n}{n+1} <1$$

Hence the series (42) converges to a positive real constant B_3 . Therefore from (41), we get

$$M(r,g)\le B_1 r^{n0}+B_2 r^{N(r)}+B_3$$

Since $N(r)\to\infty$ as $r\to\infty$ so we can write above inequality as

$$\text{Log } M(r,g)\le[1+o(1)]N(r) \log r$$

$$\le[1+o(1)][\rho\alpha^{-1}\{\eta'(\alpha[\frac{\rho}{\rho-1}\log\{(n+1)r])^{\rho-1} \}] \log r$$

$$\le[1+o(1)][\rho\alpha^{-1}\{\eta'(\alpha[\frac{\rho}{\rho-1}\log\{(n+1)r])^{\rho-1} \}]\times \alpha^{-1}\{(\alpha[\frac{\rho}{\rho-1} \log\{(n+1)r\}])^{\rho-1}\}$$

$$\le[1+o(1)]\rho\alpha^{-1}\{(\eta'+1)(\alpha[\frac{\rho}{\rho-1} \log\{(n+1)r\}])^{\rho-1}\}$$

$$\alpha[\log M(r,g)]\le(\eta'+1)\alpha([\frac{\rho}{\rho-1}\log(n+1)r\}])^{\rho-1} [1+o(1)]$$

Or $$\frac{\alpha[\log M(r,g)]}{(\alpha[\frac{\rho}{\rho-1}\log\{(n+1)r\}])^{\rho-1}} \le(\eta'+1)+[1+o(1)]$$

$$\frac{\alpha[\log m(r,g)]}{(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right])^{\rho}} \le (\eta'+1)[1+o(1)]$$

Or $$\frac{\alpha[\log m(r,g)]}{(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right])^{\rho}} \le \frac{\alpha[\log m(r,g)]}{(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right])^{\rho-1}} \le (\eta'+1)[1+o(1)]$$

$$\frac{\alpha[\log m(r,g)]}{(\alpha[\log r])^{\rho}} \le \frac{\alpha[\frac{\rho}{\rho-1}\log\{(n+1)\}]^{\rho}}{(\alpha[\log r])^{\rho}} \quad (\eta'+1)[1+o(1)]$$

$$\frac{\alpha[\log m(r,g)]}{(\alpha\left[\frac{\rho}{\rho-1}\log\{(n+1)r\}\right])^{\rho}} \le \frac{\alpha[o(1)\log\{(n+1)\}]^{\rho}}{(\alpha[\log r])^{\rho}} \quad (\eta'+1)[1+o(1)]$$

Proceeding to limits as $r\to\infty$ and using properties of $\alpha(x)$,we get

$$\sigma \leq \eta' + 1$$

Since $\varepsilon > 0$ is arbitrarily small ,we finally get

$$\sigma - 1 \leq \eta \qquad ...(43)$$

Combining (36) and (43) ,we get (28).hence theorem 2 is proved .

4-Conclution

In the present paper we generalized order ρ (α , g) and generalized type $\sigma(\alpha, g)$ the of slow growth of entire special monogenic function, and we continue the work of Susheel Kumar [7] .

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عن الرتبة والنوع لدوال تحليليه ممثله بواسطة دوال خاصة أحاديه المنشأ

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الخلاصة:-

في المقالة المعروضة درسنا أعمام الرتبة وأعمام النوع لدوال الخاصة أحادية المنشأ ذات النمو البطيء ، والوصف لأعمام الرتبة وأعمام النوع قد تم الحصول عليه بدلالة معاملات سلسلة تايلور.