On the bi-extended eigenvalues and bi-extended eigenvectors

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الملخص

تركز هذاالبحث حول مفاهيم القيم الذاتية الموسعة الثنائية و المتجهات الذاتية الموسعة الثائية. سوف نتحرى حول العدد المعقد λ المعطاه بحيث أن A و λB مؤثر متشابه او شبه متشابة حيث A, B مؤثران مقيدان معرفة على فضاء هيلبرت $\mathcal{H}$.

Abstract

This paper focuses on the concepts of bi-extended eigenvalues and bi-extended eigenvectors. It investigates the complex number λ that makes A and λB Similar or Quasi-Similar operators where A, B are bounded linear operators defined on Hilbert space $\mathcal{H}$.

Keywords: bi-extended eigenvalue, bi-extended eigenvector, Similar operator, Quasi-Similar operator.

1. Introduction and terminology

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For any operator A in $\mathcal{B}(\mathcal{H})$, the spectrum of A are denoted by $\sigma(A)$. The adjoint of $T \in \mathcal{B}(\mathcal{H})$ is denoted by T^*. A complex number λ is called an extended eigenvalue of $A \in \mathcal{B}(\mathcal{H})$ if there exists a non-zero operator $X \in \mathcal{B}(\mathcal{H})$ satisfying the equation $AX = \lambda XA$. Such an operator X is called extended eigenvector corresponding to λ, for more details see [1,2,3]. The set of all extended eigenvalues of A is denoted by E(A), and the set of all extended eigenvectors of A corresponding to λ is denoted by $E_\lambda(A)$. It is clear that $E_1(A)$ hold and equal to $\{A\}'$, such that $\{A\}'$is the commutate of A. Now we define the complex number λ is bi-extended eigenvalue for the operators $A, B \in \mathcal{B}(\mathcal{H})$, if there exists a nonzero operator $X \in \mathcal{B}(\mathcal{H})$ such that $AX = \lambda XB$.

The set of all bi-extended eigenvalues denote by $E(A, B)$, the operator X is said to be bi-extended eigenvector; while the set of all bi-extended eigenvectors denote by $E_\lambda(A, B)$. It is noteworthy that if A and B are two bounded linear operators on a Hilbert space $\mathcal{H}$, then A is similar to B if there exists invertible operator $\mathrm{T} \in \mathcal{B}(\mathcal{H})$, such that $\mathrm{AT} = \mathrm{TB}$. These operators are denoted by $\mathrm{A} \sim \mathrm{B}$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $\mathrm{A} \in \mathcal{B}(\mathcal{H}_1)$ and $\mathrm{B} \in \mathcal{B}(\mathcal{H}_2)$, then A is Quasi-Similar to B if there exist two injective with dense range bounded operators T_1 from $\mathcal{H}_1$ to $\mathcal{H}_2$ and T_2 from $\mathcal{H}_2$ to $\mathcal{H}_1$, such that $\mathrm{T}_1\mathrm{A} = \mathrm{BT}_1$ and $\mathrm{AT}_2 = \mathrm{T}_2\mathrm{B}$. This is denoted by $\mathrm{A} \approx \mathrm{B}$ [6]. Now we define the set $\mathrm{S(A, B)}$ of all complex number λ, such that the operator A is similar to the operator $\lambda\mathrm{B}$, that is $\mathrm{S(A, B)} = \{\lambda \in \mathbb{C}\colon \mathrm{A} \sim \lambda\mathrm{B}\}$, if $\lambda \in \mathrm{S(A, B)}$. Then $\mathrm{S}_\lambda\mathrm{(A, B)}$ refers to the set of all invertible operators X, such that $\mathrm{AX} = \lambda\mathrm{XB}$. Also, the set $\mathrm{QS(A, B)}$ of all complex number λ is define, such that the operator A is Quasi-Similar to the operator $\lambda\mathrm{B}$. Therefore, $\mathrm{QS(A, B)} = \{\lambda \in \mathbb{C}\colon \mathrm{A} \approx \lambda\mathrm{B}\}$ if $\lambda \in \mathrm{Q}S\mathrm{(A, B)}$. Then, $\mathrm{QS}_\lambda\mathrm{(A, B)}$ refers to the set all injective and dense range operators X, such that $\mathrm{AX} = \mathrm{X}(\lambda\mathrm{B})$. This study will always assume that A, B are non-zero operators. This paper examines the sets $\mathrm{S(A, B)}$ and $\mathrm{QS(A, B)}$, the relations between these sets and the set $\mathrm{E(A, B)}$; as well as giving some properties and important results.

2. Concepts of the bi-extended eigenvalues and bi-extended eigenvectors.

This section defines the concepts of the bi-extended eigenvalues and bi-extended eigenvectors.

Definition (2.1): We say that the complex number λ is bi-extended eigenvalue for the operators A and $B \in \mathcal{B}(\mathcal{H})$, if there exists a nonzero operator $X \in \mathcal{B}(\mathcal{H})$, such that

$$AX = \lambda XB \qquad (1)$$

The set of all bi-extended eigenvalues is denoted by $E(A, B)$, the operator X is said to be bi-extended eigenvector; while the set of all bi extended eigenvectors is denoted by $E_\lambda(A, B)$, that is,

$$E(A, B) = \{\lambda \in \mathbb{C}\colon \text{there exists a nonzero operator } X \text{ satisfying } AX = \lambda XB\}$$

$$E_\lambda(A, B) = \{X \in \mathcal{B}(\mathcal{H})\colon X \neq 0 \text{ and } AX = \lambda XB\}$$

Proposition (2.2) : Let A and $B \in \mathcal{B}(\mathcal{H})$. Then, $\tilde{E}_\lambda(A,B) = E_\lambda(A,B) \cup \{0\}$ is closed linear subspace of $\mathcal{B}(\mathcal{H})$.

Proof: First, we can prove that $\tilde{E}_\lambda(A,B)$ is linear subspace of $\mathcal{B}(\mathcal{H})$. Suppose that $T_1, T_2 \in \tilde{E}_\lambda(A,B)$, and $\alpha, \beta \in \mathbb{C}$, then $AT_1 = \lambda T_1 B$ and $AT_2 = \lambda T_2 B$. Hence, $A(\alpha T_1 + \beta T_2) = (\alpha AT_1 + \beta AT_2) = (\alpha\lambda T_1 B + \beta\lambda T_2 B) = \lambda(\alpha T_1 + \beta T_2)B$. Therefore, $\alpha T_1 + \beta T_2 \in \tilde{E}_\lambda(A,B)$. Now, it shall be assume that $T_n \in \tilde{E}_\lambda(A,B)$ for each positive integer number n, such that $T_n \to T$. Then $AT_n \to AT$ and $\lambda T_n B \to \lambda TB$, since $AT_n = \lambda T_n B$ for every n, then $AT = \lambda TB$. Thus $T \in \tilde{E}_\lambda(A,B)$. Then, $\tilde{E}_\lambda(A,B)$ is closed linear subspace of $\mathcal{B}(\mathcal{H})$. ■

Theorem (2.3) [2]: If A and B are two operators on Hilbert space $\mathcal{H}$, such that $\sigma(A) \cap \sigma(B) = \phi$, then $X = 0$ is the only solution to the operator equation $AX - XB = 0$.

Proposition (2.4): For any two operators, $A, B \in \mathcal{B}(\mathcal{H})$, there is
$\mathrm{E}(A,B) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda B) \neq \phi\}$.

Proof: Suppose that $\lambda \in \mathrm{E}(A,B)$. Then, there exists a nonzero operator $X \in \mathcal{B}(\mathcal{H})$, such that $AX = \lambda XB$. Therefore $\sigma(A) \cap \sigma(\lambda B) \neq \phi$,by theorem (2.3). Hence, $\lambda \in \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda B) \neq \phi\}$. Thus $\mathrm{E}(A,B) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda B) \neq \phi\}$.■

Example (2.5): If U is Unilateral shift operator and T has dense range, then the only solution of $UX = \lambda XT$ is $X = 0$.

Solution: It is clear that if $\lambda = 0$, then $UX = 0$ solution $BUX = 0$, thus $X = 0$. So, assume that $\lambda \neq 0$ and $UX = \lambda XT$. Then $X^* U^* = \bar{\lambda} T^* X^*$. Let $\{e_n\}_{n=0}^{\infty}$ be the usual orthonormal basis. Hence, $Ue_i = e_{i+1}$ and $U^* e_{i+1} = e_i$ for every $i = 1, 2, \dots, U^* e_1 = 0$. Now, since T has dense range, then $T^*$is injective. So, $X^* U^*(e_0) = \bar{\lambda} T^* X^*(e_0)$, yields $0 = \bar{\lambda} T^* X^*(e_0)$, since $\lambda \neq 0$ and $T^*$is injective, then $X^*(e_0) = 0$ and $X^* U^*(e_1) = \bar{\lambda} T^* X^*(e_1)$, so that $X^*(e_0) = \bar{\lambda} T^* X^*(e_1)$. Therefore, $X^*(e_1) = 0$. By employing a similar manner and using the mathematical induction, there is $X^*(e_n) = 0$, for each n. Then, $X = 0$. ■

Proposition (2.6): Suppose that A and B are two operators on finite dimensional space $\mathcal{H}$:

1- If A and B are not invertible, then $\mathrm{E}(A,B) = \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda B) \neq \phi\} = \mathbb{C}$.
2- If A and B are invertible, then $\mathrm{E}(A,B) = \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda B) \neq \phi\}$.

Proof: The case when A and B are not invertible is considered firstly. In this case, both A and B^* have non-trivial kernels. Let $\acute{X}: \text{Ker}(B^*) \longrightarrow \text{Ker}(A)$ be a nonzero operator. Define $X = \acute{X}\mathcal{P}$, where $\mathcal{P}$ denotes the orthogonal projection on kernel of B^*. Clearly, $X \neq 0$. Note further that $AX = 0$, since for every $f \in \mathcal{H}$, there is $A(\acute{X}\mathcal{P}(f)) = 0$. By defining $\acute{X}$, there is $\acute{X}\mathcal{P}(f) \in \text{Ker}(A)$ for each $f \in \mathcal{H}$, there is $AX = 0$ and $XB = 0$. In other words, $\acute{X}\mathcal{P}\big(B(f)\big) = 0$, since $\acute{X}\mathcal{P}\big(B(f)\big)$ for each $f \in \mathcal{H}$.
$B(f) \in Rang(B) = (\text{Ker}(B^*))^{\perp}$. Then, $B(f) \in (\text{Ker}(B^*))^{\perp}$, but

$$\boldsymbol{\mathcal{P}}\big(\boldsymbol{B}(\boldsymbol{f})\big) = \begin{Bmatrix} \mathbf{0} & \textbf{if } \boldsymbol{B}(\boldsymbol{f}) \notin \textbf{Ker}(\boldsymbol{B}^*) \\ \boldsymbol{B}(\boldsymbol{f}) & \textbf{if } \boldsymbol{B}(\boldsymbol{f}) \in \textbf{Ker}(\boldsymbol{B}^*) \end{Bmatrix} \qquad \textbf{(2)}$$

If $B(f) \notin \text{Ker}(B^*)$, then $\mathcal{P}\big(B(f)\big) = 0$ is hold. If $B(f) \in \text{Ker}(B^*)$; therefore, $B(f) \in \text{Ker}(B^*) \cap (\text{Ker}(B^*))^{\perp} = \{0\}$. Then, $\mathcal{P}\big(B(f)\big) = 0$. Since $\acute{X}$ is a nonzero linear operator, then $\acute{X}(0) = 0$; thus, $XB = 0$. Hence, $AX = \lambda XB$ for any $\lambda \in \mathbb{C}$. Consequently, $\text{E}(A, B) = \mathbb{C}$. Since A and B are not invertible for any complex number $\lambda, 0 \in \sigma(A) \cap \sigma(\lambda B)$, thus $\{\lambda \in \mathbb{C}: \sigma(A) \cap \sigma(\lambda B) \neq \phi\} = \mathbb{C}$.
Secondly, the study considers the case when A and B are invertible so that $0 \notin \sigma(A) \cap \sigma(\lambda B)$. To show that $\{\lambda \in \mathbb{C}: \sigma(A) \cap \sigma(\lambda B) \neq 0\} \subseteq \text{E}(A, B)$, then suppose that β is a (necessarily non-zero) complex number such that $\beta \in \sigma(A)$ and $\beta \in \sigma(\lambda B)$. Since $\beta \in \sigma(A)$, then there exists a vector u such that $Au = \beta u$. On the other hand, $\beta \in \sigma(\lambda B)$ which implies that $\lambda \neq 0$; so, ${}^{\beta}/_{\lambda} \in \sigma(B)$ and $\overline{({}^{\beta}/_{\lambda})} \in \sigma(B^*)$ as well as there is a vector v such that $B^*v = \overline{({}^{\beta}/_{\lambda})}v$. Let $X = u \otimes v$, then $AX = \lambda XB$. So, for every $f \in \mathcal{H}$. Then, $AXf = A(u \otimes v)f = (A(f, v)u) = (f, v)Au = \beta(f, v)u$ and $\lambda XBf = \lambda\big((u \otimes v)B\big)(f)) = \lambda(B(f), v)u = \lambda(f, B^*v)u =$
$\lambda(f, \overline{\left({}^{\beta}/_{\lambda}\right)} v)u = \beta(f, v)u$; consequently, $\beta \in \text{E}(A, B)$. ■

The Similar and Quasi-Similar on $\text{E}(A, B)$ will be define using the same way of $\text{E}(A, B)$.

Definition (2.7): Suppose that $A, B \in \mathcal{B}(\mathcal{H})$, then:
$S(A, B) = \{\lambda \in \mathbb{C}: A \textit{ is} \text{ Similar to } \lambda B\}$.
$QS(A, B) = \{\lambda \in \mathbb{C}: A \text{ is Quasisimilar to } \lambda B\}$.

One can prove easily the following remark:

Remark (2.8): Suppose that A and B are nonzero operators in $\mathcal{B}(\mathcal{H})$. Then:

1- $S(0,0) = \mathbb{C}$, $S(A,0) = \emptyset$ and $S(0,B) = \{0\}$.

2- $S(\alpha I, I) = \{\alpha\}$; also if $\alpha \neq 0$, then $S(I, \alpha I) = \{1/\alpha\}$, where I is the identity operator.

3- $S(\alpha A, B) = S(A, 1/\alpha\, B)$ for every nonzero complex number α.

4- $S(A,B) \subseteq QS(A,B) \subseteq E(A,B)$.

Proof: (1) It is clarified in definition (2.7).

(2) There is $S(\alpha I, I) = \{\lambda \in \mathbb{C}: \alpha I \text{ is Similar to } \lambda I\} = \{\alpha\}$. By the same way, it has been proved that $S(I, \alpha I) = \{1/\alpha\}$, where $\alpha \neq 0$.

(3) Using the same definition, $S(\alpha A, B) = S(A, 1/\alpha\, B)$ for every nonzero complex number α.

(4) Using the same way in [7], and since every invertible is injective, this is also nonzero; we have $S(A,B) \subseteq QS(A,B) \subseteq E(A,B)$. ■

Based on Remark (2.8), $S(A,B)$ is not necessary equal to $S(B,A)$.

Theorem (2.9): Suppose that A and B are two bounded operators such that A or A^* is injective and $\lambda^n = 1$ for some positive integer number n, $\lambda \neq 1$. If $AX = \lambda XB$ (Brief $X \in \widetilde{E}_\lambda(A,B)$), then the operators

$$Y_i = \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j-1} X B^j, i = 0,1, \dots, n-1$$

are the unique operators that satisfy $AY_i = \lambda^i Y_i A$, $i = 1, 2, \dots, n-1$.

Proof: Suppose that A or A^* is injective and $\lambda^n = 1$ for some positive integer number n, $\lambda \neq 1$. So if $X \in \widetilde{E}_\lambda(A,B)$, then $AX = \lambda XB$. So, $A^n X = \lambda^n X B^n$ and $A^n X = X B^n$. Let $Y_i = \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j-1} X B^j$, then $AY_i = \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j} X B^j = A^n X + \sum_{j=1}^{n-1} \lambda^{ij} A^{n-j} X B^j$

$$= XB^n + \sum_{j=1}^{n-1} \lambda^{ij} A^{n-j} X B^j = \sum_{k=0}^{n-1} \lambda^{i(k+1)} A^{n-k-1} X B^{k+1} = \left(\sum_{k=0}^{n-1} \lambda^{ik} A^{n-k-1} X B^k\right)(\lambda^i B) = \lambda^i Y_i B$$

Since $\lambda^n = 1$, $\lambda \neq 1$. Then $1 - \lambda^n = 0$. So, $(1-\lambda)(1 + \lambda + \cdots + \lambda^{n-1}) = 0$. Generally $\sum_{i=0}^{n-1} \lambda^{ij} = 0$, when $j \neq 0$ and the sum is n when $j = 0$.

$$\sum_{i=0}^{n-1} \mathrm{Y_i} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \lambda^{\mathrm{ij}} A^{n-j-1} X B^j = \sum_{j=0}^{n-1} A^{n-j-1} X B^j \sum_{i=0}^{n-1} \lambda^{\mathrm{ij}} = nA^{n-1}X$$

Now, suppose that $Z_0, Z_1, \dots, Z_{n-1}$ are operators such that $nA^{n-1}X = \sum_{i=0}^{n-1} \mathrm{Z}_i$ and $AZ_i = \lambda^{\mathrm{i}} Z_i B$, for each i, there is $nA^{n-1}\mathrm{Y_i} = \sum_{j=0}^{n-1} \lambda^{\mathrm{ij}} A^{n-j-1}(nA^{n-1}X)B^j = \sum_{j=0}^{n-1} \lambda^{\mathrm{ij}} A^{n-j-1}(\sum_{k=0}^{n-1} \mathrm{Z}_k)B^j$ $= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \lambda^{\mathrm{ij}} A^{n-j-1} B^j \lambda^{-\mathrm{ki}} \mathrm{Z}_k$, when $B = A$, then $nA^{n-1}\mathrm{Y_i} = nA^{n-1}\mathrm{Z}_i$.

If A is injective, then $A^{n-1}\mathrm{Y_i} = A^{n-1}\mathrm{Z}_i$ which implies that $\mathrm{Y_i} = \mathrm{Z}_i$. If A^* is injective, then A^{n-1} has dense range and $\mathrm{Y_i}A^{n-1} = \lambda^{-\mathrm{i(n-1)}} A^{n-1}\mathrm{Y_i} = \lambda^{-\mathrm{i(n-1)}} A^{n-1}\mathrm{Z}_i = A^{n-1}\mathrm{Z}_i$ which implies that $\mathrm{Y_i} = \mathrm{Z}_i$. ■

The following example shows that $E(A,B) \neq S(A,B)$.

Example (2.10): Let $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, then:

$AX = \alpha XB$. So that $\alpha \in E(A,B)$ and $\alpha \notin S(A,B)$.

3. Some Properties of bi-extended eigenvalues and bi-extended eigenvectors

This section studies the bi-extended eigenvalues and bi-extended eigenvectors for the operators A and B when A is Similar (Quasi-Similar) to B.

Proposition (3.1): Suppose that $A, B \in \mathcal{B}(\mathcal{H})$, then:

1- If $A \sim B$, then $\mathrm{E}(A,B) = E(A) = E(B)$.
2- If $A \approx B$, then $\mathrm{E}(A,B) = \mathrm{E}(B,A)$.
3- If $\lambda \neq 0$, then $\mathrm{E}(A,B) = \mathrm{E}(I - A, {}^{1}/_{\lambda} I - B)$, where I is the identity operator.

Proof: (1) It is clear that $E(A) = E(B)$ from [7]. Therefore, it is enough to prove $\mathrm{E}(A,B) = E(A)$.

Suppose that $\lambda \in \mathrm{E}(A,B)$, then there exists a nonzero operator X such that $AX = \lambda XB$. Since A is similar to B. Then there exists an invertible operator T such that $B = T^{-1}AT$, there is $AX = \lambda X(T^{-1}AT)$. So $A(XT^{-1}) = \lambda(XT^{-1})A$. Since $XT^{-1} \neq 0$. Then $\lambda \in \mathrm{E}(A)$. Thus $\mathrm{E}(A,B) \subseteq E(A)$.

Conversely, assume that $\lambda \in \mathrm{E}(A)$, then there exists a nonzero operator Y such that $AY = \lambda YA$.

Since A is Similar to B. Then $A = TBT^{-1}$, by substituting A on the right side of $AY = \lambda YA$, there is $AY = \lambda Y(TBT^{-1})$. So, $A(YT) = \lambda(YT)B$. Since $YT \neq 0$, there is $\lambda \in \mathrm{E}(A, B)$. So, $E(A) \subseteq \mathrm{E}(A, B)$. Then $\mathrm{E}(A, B) = E(A)$.

(2) Suppose that $\lambda \in \mathrm{E}(A, B)$, then there exists a nonzero operator X such that $AX = \lambda XB$, since A is Quasisimilar to B. Then there exists two operators Y, Z that are injective with dense range such that $AY = YB$, $ZA = BZ$. So, $AX = \lambda XB$, then $ZAX = \lambda ZXB$. Thus, $BZX = \lambda ZXB$. By multiplying both sides by Z, there is $B(ZXZ) = \lambda(ZXZ)A$, since $X \neq 0$ and Z is injective with dense range. Then $ZXZ \neq 0$. Thus, $\lambda \in \mathrm{E}(B, A)$.

Assume that $\alpha \in \mathrm{E}(B, A)$, then there exists a nonzero operator T such that $BT = \alpha TA$, since A is Quasi-similar to B. Then there exists two operators Y, Z which are injective with dense range such that $\mathrm{Y}BT = \alpha YTA$. So, $AYT = \alpha YTA$. By multiplying both sides by Y, there is $A(YTY) = \alpha(YTY)B$, since $T \neq 0$ and Y is injective with dense range, there is $YTY \neq 0$. Then $\alpha \in \mathrm{E}(A, B)$. Therefore, $\mathrm{E}(A, B) = \mathrm{E}(B, A)$.

(3) Since $\lambda \neq 0$ and by definition of $E(A, B)$, there is $AX = \lambda XB$.

So, $(I - A)X = \lambda X({}^{1}/_{\lambda} I - B)$ and $\mathrm{E}(A, B) = \mathrm{E}(I - A, {}^{1}/_{\lambda} I - B)$. ■

Corollary (3.2): Suppose that $A, B \in \mathcal{B}(\mathcal{H})$. Then:

1- If $A \sim B$, then $\mathrm{S}(A, B) = S(A) = S(B)$ and $\mathrm{QS}(A, B) = QS(A) = QS(B)$.

2- If $A \approx B$, then $\mathrm{S}(A, B) = \mathrm{S}(B, A)$ and $\mathrm{QS}(A, B) = \mathrm{QS}(B, A)$.

Some properties for the bi-extended eigenvalues are given in the following proposition.

Proposition (3.3): Suppose that $A, B \in \mathcal{B}(\mathcal{H})$. Then:

1- If $\lambda \in \mathrm{E}(A, B)$, then $\lambda^n \in \mathrm{E}(A^n, B^n)$ for every positive integer number n; also
if $\lambda \in \mathrm{S}(A, B)\big(QS(A, B)\big)$, then $\lambda^n \in \mathrm{S}(A^n, B^n)(\mathrm{QS}(A^n, B^n))$.

2- $S(B, A)^{-1} = S(A, B), QS(B, A)^{-1} = QS(A, B)$ and if A and B are an invertible operators, so
$S(B^{-1}, A^{-1}) = S(A, B)$ and $QS(B^{-1}, A^{-1}) = QS(A, B)$, then $S(B, A)^{-1} = S(B^{-1}, A^{-1}), QS(B, A)^{-1} = QS(B^{-1}, A^{-1})$.

Proof: (1) Let $\lambda \in \mathrm{E}(A,B)$, then there exists a nonzero operator T such that $AT = \lambda TB$. So, $AAT = \lambda ATB$, and $A^2T = \lambda^2 TB^2$ in general $A^nT = \lambda^n TB^n$ for every positive integer number n. Thus, $\lambda^n \in \mathrm{E}(A^n, B^n)$.

Using the same way, $\lambda \in \mathrm{S}(A,B)(QS(A,B))$ can be proved. Then $\lambda^n \in \mathrm{S}(A^n,B^n)(\mathrm{QS}(A^n,B^n))$ for every positive integer number n.

(2) $S(B,A)^{-1} = S(A,B)$ will be proven and the other one can be proved by employing the same way. Suppose that $\lambda \in \mathrm{S}(A,B)$, then there exists an invertible operator T such that $T^{-1}AT = \lambda B$. So, ${}^{1}/_{\lambda} A = (T^{-1})^{-1}BT^{-1}$ and ${}^{1}/_{\lambda} \in S(B,A)$. Then $\lambda \in S(B,A)^{-1}$. Hence $S(A,B) \subseteq S(B,A)^{-1}$. So, the prove of $S(B,A)^{-1} \subseteq S(A,B)$ is Similar. Thus $S(B,A)^{-1} = S(A,B)$. If A and B are invertible operators, then $S(B^{-1},A^{-1}) = S(A,B)$ can be proved.

Let $\lambda \in \mathrm{S}(A,B)$, then there exists an invertible operator T such that $T^{-1}AT = \lambda B$. So, $AT = \lambda TB$, since A and B are invertible, there exists $TB^{-1} = \lambda AT^{-1}$, $B^{-1}(T^{-1}) = \lambda(T^{-1})A^{-1}$ and $\lambda \in S(B^{-1},A^{-1})$. Therefore, $S(A,B) \subseteq S(B^{-1},A^{-1})$ and $S(B^{-1},A^{-1}) \subseteq S(A,B)$. Using the same way, $QS(B^{-1},A^{-1}) = QS(A,B)$ can be proved. Similarly, the Similar $S(A,B)$ can be proved. Based on $S(B,A)^{-1} = S(A,B)$ and $S(B^{-1},A^{-1}) = S(A,B)$, where A and B are an invertible operators, there exists $S(B,A)^{-1} = S(B^{-1},A^{-1})$; also, when A and B are invertible operators, there exists $QS(B,A)^{-1} = QS(B^{-1},A^{-1})$. ■

Proposition (3.4): The operators $A,B,C,D,X \in \mathcal{B}(\mathcal{H})$, then:

1- $\widetilde{\mathrm{E}}_{\alpha}(A,B)\widetilde{\mathrm{E}}_{\beta}(B,C) \subset \widetilde{\mathrm{E}}_{\alpha\beta}(A,C)$. In particular, if $A = C$, then $\widetilde{\mathrm{E}}_{\alpha}(A,B)\widetilde{\mathrm{E}}_{\beta}(B,A) \subset \widetilde{\mathrm{E}}_{\alpha\beta}(A)$.

2- $\widetilde{\mathrm{E}}_{1}(A,B)\widetilde{\mathrm{E}}_{\alpha}(B,A)\widetilde{\mathrm{E}}_{1}(A,B) \subset \widetilde{\mathrm{E}}_{\alpha}(A,B)$.

3- If $X \in \mathrm{E}_{\alpha}(A,B) \cap \mathrm{E}_{\beta}(C,D)$, then $X \in \mathrm{E}_{\alpha\beta}(CA,DB) \cap \mathrm{E}_{\alpha\beta}(AC,BD)$.

Proof: (1) Suppose that $X \in \widetilde{\mathrm{E}}_{\alpha}(A,B)$ and $Y \in \widetilde{\mathrm{E}}_{\beta}(B,C)$, then $AX = \alpha XB$, $BY = \beta YC$. So, $AXY = \alpha XBY$, $A(XY) = \alpha\beta(XY)C$ and $XY \in \widetilde{\mathrm{E}}_{\alpha\beta}(A,C)$. This is clear in certain case.

(2) Let $T \in \widetilde{\mathrm{E}}_{\alpha}(B,A)$ and $X,Y \in \widetilde{\mathrm{E}}_{1}(A,B)$, then $BT = \alpha TA$, $AX = XB$ and $AY = YA$. So, $XBTY = \alpha XTAY$, $A(XTY) = \alpha(XTY)B$ and $XTY \in \widetilde{\mathrm{E}}_{\alpha}(A,B)$. Thus, $\widetilde{\mathrm{E}}_{1}(A,B)\widetilde{\mathrm{E}}_{\alpha}(B,A)\widetilde{\mathrm{E}}_{1}(A,B) \subset \widetilde{\mathrm{E}}_{\alpha}(A,B)$.

(3) Since $X \in E_{\alpha}(A,B)$ and $X \in E_{\beta}(C,D)$, then $AX = \alpha XB$ and $CX = \beta XD$. So, $AXD = \alpha XBD$, $(AC)X = \alpha\beta X(BD)$, $X \in E_{\alpha\beta}(AC,BD)$ and $CAX = \alpha CXB$. Therefore, $(CA)X = \alpha\beta X(DB)$ and $X \in E_{\alpha\beta}(CA,DB)$. ■

Corollary (3.5): The operators $A,B,C,D,X \in \mathcal{B}(\mathcal{H})$, then:

1- $S_{\alpha}(A,B)\widetilde{E}_{\beta}(B,C) \subset \widetilde{E}_{\alpha\beta}(A,C)$ and $QS_{\alpha}(A,B)\widetilde{E}_{\beta}(B,C) \subset \widetilde{E}_{\alpha\beta}(A,C)$. In particular, if $A = C$, then $S_{\alpha}(A,B)\widetilde{E}_{\beta}(B,A) \subset \widetilde{E}_{\alpha\beta}(A)$ and $QS_{\beta}(A,B)\widetilde{E}_{\alpha}(B,A) \subset \widetilde{E}_{\alpha\beta}(A)$.

2- $S_{\alpha}(A,B)S_{\beta}(B,C) \subset S_{\alpha\beta}(A,C)$ and $S_{\alpha}(A,B)QS_{\beta}(B,C) \subset QS_{\alpha\beta}(A,C)$.

3- $S_1(A,B)S_{\alpha}(B,A)S_1(A,B) \subset S_{\alpha}(A,B)$ and $QS_1(A,B)QS_{\alpha}(B,A)QS_1(A,B) \subset QS_{\alpha}(A,B)$.

Proof: by employing the same way of proposition (3.4). ■

Theorem (3.6): Suppose that $A \approx \tilde{A}$ and $B \approx \tilde{B}$, then $E(A,B) = E(\tilde{A},\tilde{B})$.

Proof: Suppose that $A \approx \tilde{A}$ and $B \approx \tilde{B}$, then there exists S,T,Z,W that are injective with dense range such that $AS = S\tilde{A}$, $TA = \tilde{A}T$, $BZ = Z\tilde{B}$, $WB = \tilde{B}W$. Now, let $\lambda \in E(A,B)$, then there exists a nonzero operator X, such that $AX = \lambda XB$. So, $TAXZ = \lambda TXBZ$ and $\tilde{A}(TXZ) = \lambda(TXZ)\tilde{B}$.
Since $X \neq 0$ and T,Z are injective with dense range, then $TXZ \neq 0$. So, $\lambda \in E(\tilde{A},\tilde{B})$ and $E(A,B) \subseteq E(\tilde{A},\tilde{B})$.

Conversely, let $\lambda \in E(\tilde{A},\tilde{B})$, then there exists a nonzero operator Y, such that $\tilde{A}Y = \lambda Y\tilde{B}$. So, $S\tilde{A}YW = \lambda SY\tilde{B}W$ and $A(SYW) = \lambda(SYW)B$. Since $Y \neq 0$ and S,W are injective with dense range. Hence, $SYW \neq 0$, that is, $\lambda \in E(A,B)$. Therefore, $E(\tilde{A},\tilde{B}) \subseteq E(A,B)$ and $E(\tilde{A},\tilde{B}) = E(A,B)$. ■

Proposition (3.7) [5]: Assume that A and B are two self adjoint operators, where A is injective; if $\lambda \in E(A,B)$, then $\lambda \in \mathbb{R}$.

Lemma (3.8): Suppose that A is invertible operator and B is nilpotent operator. Then the equation $AX = \lambda XB$ have only the zero solution.

Proof: Suppose B is nilpotent, then there exists a positive integer number n such that $B^n = 0$. Then $AX = \lambda XB$. So, $A^2X = \lambda^2 XB^2$. Hence, $A^nX = \lambda^n XB^n$ for each n, since $B^n = 0$, there is $A^nX = 0$, and A is invertible, which implies that $X = 0$. ■

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