The Solution of Faltung Type Volterra Integro-Differential Equation of First Kind using Complex SEE Transform

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Abstract
Complex SEE integral transform has been applied in solving Faltung type Volterra integro-differential equation of first kind, the capability and effectiveness of SEE transform in solving such an important equation has been demonstrated and proven through the practical application of the transform on numerical examples.

Keywords: Volterra integro- differential equation, complex SEE integral transform, Faltung, inverse complex SEE transform.

حل معادلة فولتيرا التكاملية–التفاضلية من نوع فالتونج من النوع الأول باستخدام تحويل SEE المعقد

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الخلاصة
تم تطبيق تحويل SEE المتكامل المعقد في حل معادلة فولتيرا التكاملية التفاضلية من نوع فالتونج من النوع الأول ، وقد تم إثبات قدرة وفعالية تحويل SEE في حل مثل هذه المعادلة المهمة من خلال التطبيق العملي للتحويل في الأمثلة الرياضية.

الكلمات المفتاحية: معادلة فولتيرا التكاملية التفاضلية ، تحويل SEE المتكامل المعقد ، فالتونج ، تحويل SEE المعقد العكسي.

1. Introduction

Integral transforms have a significant importance in many scientific fields, due to their ability to remapping from one domain into another, in which provides more flexibility in solving many scientifical problems.

There are many scientific and engineering fields in which Volterra integro-differential equations participated into, such as physics, astronomy, biotechnology, radiology and many other fields. For the significant importance of Volterra integral equations, many integral transforms have been proposed to solve the first and second kind Volterra integral equations, such as Laplace, Kamal, Mahgoub, Aboodh and other transforms, [1-12]. The novelty of complex SEE (Sadiq-Emad-Eman) integral transform led to the lacking of its usage in many fields, and for many applications. In this paper, the complex SEE transform is going to be used to solve Faltung type Volterra integro-differential equation of first kind with practical numerical examples on the matter.

2. Basic Concepts of Complex SEE Transformation

Complex SEE (Sadiq-Emad-Eman) transform denoted by the operator $S^c\{.\}$ defined as, [13-14]

$S^c\{f(t)\} = T(iv) = \frac{1}{v^n}\int_{t=0}^{\infty} e^{-ivt} f(t)\,dt$, $t \geq 0$, i complex number , $l_1 \leq v \leq l_2$, $n \in \mathbb{Z}$ and $l_1, l_2 > 0$

The variable (iv) in this complex convert is used to factor the variable t in the argument of the function $f(t)$.

2.1. Complex SEE Transform of Some Frequently Encountered Functions, [13-14]

- $S^c\{C\} = -\frac{iC}{v^{n+1}}$, *where C is constant* .
- $S^c\{t\} = -\frac{1}{v^{n+2}}$.
- $S^c\{t^2\} = \frac{(2!)i}{v^{n+3}}$.
- $S^c\{t^3\} = \frac{(3!)}{v^{n+4}}$.
- $S^c\{e^{bt}\} = -\frac{1}{v^n}\left[\frac{b}{b^2+v^2} + i\frac{v}{b^2+v^2}\right]$, *where b is constant* .
- $S^c\{\sin(bt)\} = \frac{-b}{v^n(v^2-b^2)}$.
- $S^c\{\cos(bt)\} = \frac{-iv}{v^n(v^2-b^2)}$.
- $S^c\{\sinh(bt)\} = \frac{-b}{v^n(v^2+b^2)}$.
- $S^c\{\cosh(bt)\} = \frac{-iv}{v^n(v^2+b^2)}$.

2.2. Fundamental Properties of Complex SEE Transform, [13-14]

- Linearity: $S^c\{af_1(t) + bf_2(t)\} = aS^c\{f_1(t)\} + bS^c\{f_2(t)\}$.
- Change of scale: $S^c\{g(at)\} = \frac{1}{a^{n+1}} T\left(\frac{iv}{a}\right)$.
- Shifting: $S^c\{e^{at}f(t)\} = \frac{(iv-a)^n}{v^n} T(iv-a)$.
- First derivative: $\frac{-f(0)}{v^n} + iv\,T(iv) = S^c\{f'(t)\}$.
- Second derivative: $\frac{-f'(0)}{v^n} - \frac{if(0)}{v^{n-1}} - v^2\,T(iv) = S^c\{f''(t)\}$.
- m^{th} derivative: $\frac{1}{v^n}\left[-f^{(m-1)}(0) - ivf^{(m-2)}(0) - (iv)^2 f^{(m-3)}(0) - \cdots - (iv)^{m-1}f(0)\right] + (iv)^m T(iv) = S^c\{f^{(m)}(t)\}$.
- Convolution: $S^c\{g(t) * h(t)\} = v^n\, T_1(iv).T_2(iv)$.

3. Complex SEE Transform for Solving Faltung Type Volterra Integro-Differential Equation of First Kind

Faltung type Volterra integro- differential equation of first kind is defined by:

$$\left.\begin{array}{c}\int_{u=0}^{t} k_1(t-u)\, w(u)du + \int_{u=0}^{t} k_2(t-u)\, w^{(m)}(u)du = g(t) \\ k_2(t-u) \neq 0\end{array}\right\} \quad \dots (1)$$

With: $$\left.\begin{array}{c} w(0) = \delta'_0 ,\ w'(0) = \delta_1 ,\ w''(0) = \delta_2 , \dots , \\ w^{(m-1)}(0) = \delta_{m-1}\end{array}\right\} \quad \dots (2)$$

Where:

$k_1(t-u)$ and $k_2(t-u) \equiv$ Faltung type kernel of integral equation.

$w(t) \equiv$ Unknown function of t.

$w^{(m)}(t) \equiv m^{th}$ derivative of unknown function.
$g(t) \equiv$ known function of t.
$\delta_0, \delta_1, \delta_2, \dots, \delta_{m-1} \equiv$ real numbers.
Taking complex SEE transformation of both sides of equation (1): $S^c\left\{\int_{u=0}^{t} k_1(t-u)\, w(u)du\right\} + S^c\left\{\int_{u=0}^{t} k_2(t-u)\, w^{(m)}(u)du\right\} = S^c\{g(t)\} \quad \dots (3)$

Applying convolution property of complex SEE integral transform on equation (3):
$v^n\, S^c\{k_1(t)\}. S^c\{w(t)\} + v^n\, S^c\{k_2(t)\}. S^c\{w^{(m)}(t)\} = S^c\{g(t)\} \quad \dots (4)$

Now, applying "Complex SEE Transform property" of derivative of functions on equation (4):
$v^n\, S^c\{k_1(t)\}. S^c\{w(t)\} + v^n\, S^c\{k_2(t)\}. \left[\frac{1}{v^n}\left(-w^{(m-1)}(0) - iv\, w^{(m-2)}(0) - (iv)^2\, w^{(m-3)}(0) - \dots - (iv)^{m-1} w(0)\right) + (iv)^m T(iv)\right] = S^c\{g(t)\} \quad \dots (5)$

Substituting equation (2) in equation (5): $v^n\, S^c\{k_1(t)\}. S^c\{w(t)\} + v^n\, S^c\{k_2(t)\}. \left[\frac{1}{v^n}(-\delta_{m-1} - iv\, \delta_{m-2} - (iv)^2\, \delta_{m-3} - \dots - (iv)^{m-1}\delta_0) + (iv)^m T(iv)\right] = S^c\{g(t)\}$

$\Rightarrow [v^n\, S^c\{k_1(t)\} + (iv)^m v^n\, S^c\{k_2(t)\}] S^c\{w(t)\} = S^c\{g(t)\} + v^n\, S^c\{k_2(t)\}\left(\frac{\delta_{m-1}}{v^n} + iv\frac{\delta_{m-2}}{v^n} + \dots + \frac{(iv)^{m-1}}{v^n}\delta_0\right)$.

$\Rightarrow [v^n\, S^c\{k_1(t)\} + (iv)^m v^n\, S^c\{k_2(t)\}] S^c\{w(t)\} = S^c\{g(t)\} + S^c\{k_2(t)\}(\delta_{m-1} + iv\delta_{m-2} + \dots + (iv)^{m-1}\delta_0)$.

$\Rightarrow S^c\{w(t)\} = \frac{S^c\{g(t)\} + S^c\{k_2(t)\}(\delta_{m-1} + iv\delta_{m-2} + \dots + (iv)^{m-1}\delta_0)}{v^n\, [S^c\{k_1(t)\} + (iv)^m v^n\, S^c\{k_2(t)\}]} \quad \dots (6)$

And $S^c\{k_1(t)\} + (iv)^m\, S^c\{k_2(t)\} \neq 0$.

Taking inverse complex SEE integral transform on equation (6), gives the required exact solution of Faltung type Volterra integrao- differential equation of first kind, which is given by equation (1) with conditions in equation (2).

4. Numerical Problems

In this section, some numerical problems are considered to clarified complex SEE methodology in solving Faltung type Volterra integrao- differential equation of first kind.

Problem (4.1): Consider the following equation: $\int_{u=0}^{t} w(u).(t-u)\, du + \int_{u=0}^{t} w'(u)\, (t-u)^2\, du = 3(t - \sin(t)) \quad \dots (7)$
With: $w(0) = 0 \quad \dots (8)$
Applying complex SEE integral transform on equation (7): $S^c\left\{\int_{u=0}^{t} w(u)\, (t-u)\, du\right\} + S^c\left\{\int_{u=0}^{t} w'(u)\, (t-u)^2\, du\right\} = 3S^c\{t\} - 3S^c\{\sin(t)\} \quad \dots (9)$

Apply convolution and complex SEE transform properties of derivative of function on equation (9):
$v^n\left(\frac{-1}{v^{n+2}}\right) S^c\{w(t)\} + v^n\left(\frac{(2!)\, i}{v^{n+3}}\right)\left[\frac{-w(0)}{v^n} + iv\, S^c\{w(t)\}\right] = 3\left(\frac{-1}{v^{n+2}}\right) + \frac{3}{v^n(v^2-1)} \quad \dots (10)$
Now, using equation (8) in equation (10): $\frac{-1}{v^2}\, S^c\{w(t)\} + \frac{2i}{v^3}\, [iv\, S^c\{w(t)\}] = \frac{-3}{v^{n+2}} + \frac{3}{v^n(v^2-1)}$,

$\Rightarrow \frac{-1}{v^2}\, S^c\{w(t)\} - \frac{2}{v^2}\, S^c\{w(t)\} = \frac{-3}{v^{n+2}} - \frac{3}{v^n(v^2-1)}$,

$\Rightarrow \frac{-3}{v^2}\, S^c\{w(t)\} = \frac{-3}{v^{n+2}} + \frac{3}{v^n(v^2-1)}$ ... (11)

$\Rightarrow S^c\{w(t)\} = \frac{1}{v^n} - \frac{v^2}{v^n(v^2-1)} = \frac{1}{v^n}\left[\frac{v^2-1-v^2}{(v^2-1)}\right], \quad \Rightarrow S^c\{w(t)\} = \frac{-1}{v^n(v^2-1)}$ ... (12)

Taking inverse complex SEE integral transform to equation (12), the required exact solution of equation (7) with equation (8) would be: $w(t) = \sin(t)$.

Problem (4.2): Consider the following equation: $\int_{u=0}^{t} w(u)\cos(t-u)\,du + \int_{u=0}^{t} w'''(u)\sin(t-u)\,du = \sin(t) - \cos(t) + 1$ ... (13)

With $w(0) = w'(0) = 1$ and $w''(0) = -1$ ... (14)

Taking complex SEE integral transform to both sides of equation (13): $S^c\left\{\int_{u=0}^{t} w(u)\cos(t-u)\,du\right\} + S^c\left\{\int_{u=0}^{t} \sin(t-u)\; w'''(u)du\right\} = S^c\{\sin(t)\} - S^c\{\cos(t)\} + S^c\{1\}$ ... (15)

Applying convolution property of complex SEE transform on equation (15):

$v^n S^c\{w(t)\}.\, S^c\{\cos(t)\} + v^n S^c\{w'''(t)\}\, S^c\{\sin(t)\} = S^c\{\sin(t)\} - S^c\{\cos(t)\} + S^c\{1\}$ ... (16)

Applying the property complex SEE transform of derivative of functions on equation (16):

$v^n S^c\{w(t)\}\left(\frac{-iv}{v^n(v^2-1)}\right) + v^n\left(\frac{-1}{v^n(v^2-1)}\right)\left[\frac{1}{v^n}\left(-w''(0) - iv\, w'(0) - (iv)^2 w(0)\right) + (iv)^3 S^c\{w(t)\}\right] = \frac{-1}{v^n(v^2-1)} + \frac{iv}{v^n(v^2-1)} - \frac{i}{v^{n+1}}$,

$\Rightarrow \frac{-iv}{(v^2-1)} S^c\{w(t)\} - \frac{1}{(v^2-1)}\left[\frac{1}{v^n}\left(-w''(0) - iv\, w'(0) - (iv)^2 w(0)\right) + (iv)^3 S^c\{w(t)\}\right] = \frac{-1}{v^n(v^2-1)} + \frac{iv}{v^n(v^2-1)} - \frac{i}{v^{n+1}}$ ... (17)

Using equation (14) in equation (17):

$\frac{-iv}{(v^2-1)} S^c\{w(t)\} - \frac{1}{(v^2-1)}\left[\frac{1}{v^n}(1 - iv - (iv)^2) + (iv)^3 S^c\{w(t)\}\right] = \frac{-1}{v^n(v^2-1)} + \frac{iv}{v^n(v^2-1)} - \frac{i}{v^{n+1}}$,

$\Rightarrow \left[\frac{-iv}{(v^2-1)} - \frac{(iv)^3}{(v^2-1)}\right] S^c\{w(t)\} = \cancel{\frac{-1}{v^n(v^2-1)}} + \cancel{\frac{iv}{v^n(v^2-1)}} - \frac{i}{v^{n+1}} + \cancel{\frac{1}{v^n(v^2-1)}} - \cancel{\frac{iv}{v^n(v^2-1)}} - \frac{(iv)^2}{v^n(v^2-1)}$

$\Rightarrow \frac{-iv-(iv)^3}{(v^2-1)} S^c\{w(t)\} = \frac{-i}{v^{n+1}} - \frac{(iv)^2}{v^n(v^2-1)}$, $\Rightarrow S^c\{w(t)\} = \frac{-i(v^2-1)}{v^{n+1}(-iv-(iv)^3)} - \frac{(iv)^2(v^2-1)}{v^n(v^2-1)(-iv-(iv)^3)}$,

$\Rightarrow S^c\{w(t)\} = \frac{-i\cancel{(v^2-1)}}{v^{n+1}\, iv\,\cancel{(-1-(iv)^2)}} - \frac{(iv)^2}{v^n(iv)[-1-(iv)^2]}$, $\Rightarrow S\{w(t)\} = \frac{-1}{v^{n+2}} - \frac{iv}{v^n(v^2-1)}$ ... (18)

Taking inverse complex SEE integral transform to both sides of equation (18), the required exact solution of equation (13) with equation (14) would be: $w(t) = t + \cos(t)$.

5. Conclusions

In this work, the authors succussed in discussing the applicability of complex SEE transform in solving Faltung type Volterra integro- differential equation of first kind, and the complet solution methodology have been explained through practical numerical problems. The concluded results from the discussed problems showed the effectiveness of complex SEE transform in finding the required exact solution of Faltung type Volterra integro- differential equation of first kind.

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