Extended B-spline approach by using continuous least square approximation

Dr.Saad Shakir Mahmoud & Noor Qasim Ali
AL-Mustansiriya University/College of Education/Department of Mathematics
SaadShakirMahmoud@Yahoo.com

Abstract

Boundary value problem of two order ordinary differential equations is solved using a new approach of extended B-spline interpolation method by using continuous least square approximation. There is one free scalar ($\lambda \in R^{n+1}$) that control the polynomial of approximation solution. This method produces better results than classical B-spline interpolation method.
Keywords: cubic B-spline, boundary value problem, least square approximation, ordinary differential equations, Extended B-spline, extended cubic B-spline.

1. Introduction

Ordinary Differential Equations (ODE) have a long story applied in many all fields. A numerical solution of (ODE) have made great expansion over many centuries', and There have been emerged numerous new ideas in addition to numerous complex methods for solving (ODE), in order that the numerical methods for solving (ODE) have been heightened, systems of (ODE) has been applied to many problems in engineering, physics, biology and so on.[4]

Let $S = \{x_0, \dots, x_n\}$ be a set at partition of $[0,1]$.

The zero degree B-spline is defined as follows:

$$B_{t,0} = \begin{cases} 1 & x \in [x_t, x_{t+1}) \\ 0 & other\ wise \end{cases}$$

And for positive r, it is defined in the following recursive form: [6]

$$B_{t,r} = \left(\frac{x - x_t}{x_{t+r} - x_t}\right) B_{t,r-1}(x) + \left(\frac{x_{t+r+1} - x}{x_{t+r+1} - x_{t+1}}\right) B_{t+1,r-1}(x)$$

Where $t = 3$ and $r = 0$, we can get the cubic B-spline is defined as follows:

$$B_{3,0} = \frac{1}{h^3}\begin{cases} (x - x_{-2})^3, & x \in [x_{-2}, x_{-1}) \\ (x - x_{-2})^3 - 4(x - x_{-1})^3, & x \in [x_{-1}, x_0) \\ (x_2 - x)^3 - 4(x_1 - x)^3, & x \in [x_0, x_1) \\ (x_2 - x)^3, & x \in [x_1, x_2) \end{cases}$$

2. New approach for solving ODE by cubic b-spline

The two point boundary value problems as the following

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \;\; 0 \le x \le 1 \qquad \ldots (1)$$

$$y(a) = Z_1, \qquad y(b) = Z_2$$

Where $p(x), q(x), f(x)$ are given functions, and $p(x), q(x)$ are continuous.

Let

$$Y(x) = \sum_{k=-1}^{n+1} c_k B_{3,k}(x)$$

Be an approximate solution of Eq.(1), where c_k is unknown real coefficient and $B_{3,k}(x)$ are cubic B-spline functions. [9]

Let $x_0, x_1, \ldots, x_n$ are n+1 grid points in the interval$[a, b]$, and also $x_i = a + ih, \; i = 0,1, \ldots, n, \; x_0 = a, x_n = b, and\; h = (b - a)/n.$ [1]

In order to get a matrix of transactions are contrary to the matrix from behind to become the image on the form follows:

B-spline	x_{-2}	x_{-1}	x_0	x_1	x_2
B	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}
B'	u_{21}	u_{22}	u_{23}	u_{24}	u_{25}
B"	u_{31}	u_{32}	u_{33}	u_{34}	u_{35}

Table (1): derivative cubic B-spline

We can formulate the matrix coefficient by rearrange the matrix of derivative.

$$d_i(x_i) = u_{34} + p(x_i)u_{24} + q(x_i)u_{14}$$

$$e_i(x_i) = u_{33} + p(x_i)u_{23} + q(x_i)u_{13}$$

$$v_i(x_i) = u_{32} + p(x_i)u_{22} + q(x_i)u_{12}$$

In addition to the matrix is an $(n + 3) \times (m + 3)$ dimensional given by:

$$A = \begin{bmatrix} u_{12} & u_{13} & u_{14} & 0 & 0 & \cdots & 0 & 0 & 0 \\ d_i & e_i & v_i & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d_i & e_i & v_i & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & d_i & e_i & v_i \\ 0 & 0 & 0 & 0 & 0 & \cdots & u_{12} & u_{13} & u_{14} \end{bmatrix}$$

This system can be written in the matrix-vector form as follows: [4]

$$\mathrm{AB} = \mathrm{F}$$

Such that

$$B = \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_{n+1} \end{bmatrix}, F = \begin{bmatrix} y(a) = z_1 \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \\ y(b) = z_2 \end{bmatrix}$$

On $[x_i, x_{i+1}]$ the solution of boundary value problem is given by

$$Y_i = \frac{1}{h^3}[c_{i-1}(x_{i+1} - x)^3 + c_i((x_{i+1} - x)^3 - 4(x_{i+1} - x)^3) + c_{i+1}((x - x_i)^3 - 4(x - x_i)^3) + c_{i+2}(x - x_i)^3], \qquad i = 0,1, \dots, n-1$$

We get $Y_i = c_{3,i}x^3 + c_{2,i}x^2 + c_{1,i}x + c_{0,i} \;\; for\; i = 0, \dots, n-1$ be the solution of the ODE by Cubic B-spline on the interval $[a, b]$.

Now we can adding the new scalar $\lambda \in R^{n+1}$ to the solution Y_i and the new form of the polynomial Y_i is given as follows:

$$Y_{i\lambda^i} = \lambda_0^i c_{3,i} x^3 + \lambda_1^i c_{2,i} x^2 + \lambda_2^i c_{1,i} x + \lambda_3^i c_{0,i} \qquad \dots (2)$$

And we can find the value of $\lambda^i = \left[\lambda_0^i; \lambda_1^i; \lambda_2^i; \lambda_3^i\right]^T$ which minimize the norm $\left\|Y_{i\lambda^i} - Y_{real}\right\|$ and to clarify that, as follows:

From Eq. (1) and Eq. (2), we get:

$$\left(y_{i\lambda^i}\right)''(x) + p(x)\left(y_{i\lambda^i}\right)'(x) + q(x)\left(y_{i\lambda^i}\right)(x) = f(x)$$

Since $y_{i\lambda^i}$ is an approximation for y_{real} .

Let $Er = Error$

By using least square approximation technique, we squared Eq. (3)

$$Er = \left[\left(y_{i\lambda^i}\right)''(x) + p(x)\left(y_{i\lambda^i}\right)'(x) + q(x)\left(y_{i\lambda^i}\right)(x) - f(x)\right] \qquad \dots (3)$$

$$Er^2 = \left[\left(y_{i\lambda^i}\right)''(x) + p(x)\left(y_{i\lambda^i}\right)'(x) + q(x)\left(y_{i\lambda^i}\right)(x) - f(x)\right]^2, \qquad in\ [x_i, x_{i+1}] \qquad \dots (4)$$

By taking interpolation for Eq. (4), we get

$$\int_{x_i}^{x_{i+1}} Er^2 dx = \int_{x_i}^{x_{i+1}} \left[\left(y_{i\lambda^i}\right)''(x) + p(x)\left(y_{i\lambda^i}\right)'(x) + q(x)\left(y_{i\lambda^i}\right)(x) - f(x)\right]^2 dx \qquad \dots (5)$$

To minimize we differentiable the both sides for Eq. (5) with respect to λ^i, and hence

$$\frac{\partial}{\partial \lambda^i} \int_{x_i}^{x_{i+1}} Er^2 dx = \int_{x_i}^{x_{i+1}} \frac{\partial}{\partial \lambda^i} \left[\left(y_{i\lambda^i}\right)''(x) + p(x)\left(y_{i\lambda^i}\right)'(x) + q(x)\left(y_{i\lambda^i}\right)(x) - f(x)\right]^2 dx = 0$$

Let

$$\left(y_{i\lambda^i}\right)''(x) + p(x)\left(y_{i\lambda^i}\right)'(x) + q(x)\left(y_{i\lambda^i}\right)(x) = w_0^i\lambda_0^i + w_1^i\lambda_1^i + w_2^i\lambda_2^i + w_3^i\lambda_3^i$$

Where

$$\int_{x_i}^{x_{i+1}} \frac{\partial}{\partial \lambda^i}\left[w_0^i\lambda_0^i + w_1^i\lambda_1^i + w_2^i\lambda_2^i + w_3^i\lambda_3^i - f(x)\right]^2 dx = 0$$

And we have the following system of linear equations:

$$\begin{bmatrix} \int_{x_i}^{x_{i+1}} w_0^{i^2} dx & \int_{x_i}^{x_{i+1}} w_0^i w_1^i dx & \int_{x_i}^{x_{i+1}} w_0^i w_2^i dx & \int_{x_i}^{x_{i+1}} w_0^i w_3^i dx \\ \int_{x_i}^{x_{i+1}} w_1^i w_0^i dx & \int_{x_i}^{x_{i+1}} w_1^{i^2} dx & \int_{x_i}^{x_{i+1}} w_1^i w_2^i dx & \int_{x_i}^{x_{i+1}} w_1^i w_3^i dx \\ \int_{x_i}^{x_{i+1}} w_2^i w_0^i dx & \int_{x_i}^{x_{i+1}} w_2^i w_1^i dx & \int_{x_i}^{x_{i+1}} w_2^{i^2} dx & \int_{x_i}^{x_{i+1}} w_2^i w_3^i dx \\ \int_{x_i}^{x_{i+1}} w_3^i w_0^i dx & \int_{x_i}^{x_{i+1}} w_3^i w_1^i dx & \int_{x_i}^{x_{i+1}} w_3^i w_2^i dx & \int_{x_i}^{x_{i+1}} w_3^{i^2} dx \end{bmatrix} \begin{bmatrix} \lambda_0^i \\ \lambda_1^i \\ \lambda_2^i \\ \lambda_3^i \end{bmatrix} = \begin{bmatrix} \int_{x_i}^{x_{i+1}} w_0^i f(x) dx \\ \int_{x_i}^{x_{i+1}} w_1^i f(x) dx \\ \int_{x_i}^{x_{i+1}} w_2^i f(x) dx \\ \int_{x_i}^{x_{i+1}} w_3^i f(x) dx \end{bmatrix}$$

$C\lambda^i = F$, and the solution of the last system is given by $\lambda^i = C^{-1}F$.

3. Numerical results

In this section we consider the following ordinary differential equation (ODE).

$$y'' + y' + y = x^2 + 3x + 4, \qquad on\ [0,1]$$

$with\ y(0) = 1,\ \ y(1) = 3\ \ \ ,\ \ for\ \ n = 10$

The exact solution $y = x^2 + x + 1$

Let $x_0 = a = 0, x_n = b = 1,$ so that $h = \frac{b-a}{n}$

$\therefore h = \frac{1-0}{10} = \frac{1}{10} = 0.1$

Since $x_0 = 0, and\ x_n = x_{n-1} + h,\ \ n = 1,2, \dots ,10$

$if\ n = 1\ then\ x_1 = x_0 + h \Rightarrow 0 + 0.1 = 0.1$

$if\ n = 2\ then\ x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

And so on, $x_3 = 0.3, x_4 = 0.4, x_5 = 0.5, x_6 = 0.6, x_7 = 0.7, x_8 = 0.8,$

$x_9 = 0.9, x_{10} = 1$

Where $p(x_i) = 1, q(x_i) = 1\ and\ h = 0.1$

$$d_i(x_i) = \frac{6}{h^2} + \frac{-3}{h} p(x_i) + (1)q(x_i) = 571$$

$$e_i(x_i) = \frac{-12}{h^2} + (0)p(x_i) + (4)q(x_i) = -1196$$

$$v_i = \frac{6}{h^2} + \frac{3}{h} p(x_i) + (1)q(x_i) = 631$$

To find $Y_i\ for\ B - spline$

$\therefore Y_0 = 1 + x + x^2 + 1.0747 \times 10^{-12} x^3$

$Y_1 = 1 + x + x^2 + 1.0974 \times 10^{-12} x^3$

$Y_2 = 1 + x + x^2 - 3.3307 \times 10^{-12} x^3$

$Y_3 = 1 + x + x^2 + 1.8874 \times 10^{-12} x^3$

$Y_4 = 1 + x + x^2 + 2.5905 \times 10^{-13} x^3$

$Y_5 = 1 + x + x^2 + 2.4376 \times 10^{-13} x^3$

$Y_6 = 1 + x + x^2 - 1.8198 \times 10^{-12} x^3$

$Y_7 = 1 + x + x^2 + 1.0550 \times 10^{-12} x^3$

$Y_8 = 1 + x + x^2$

$Y_9 = 1 + x + x^2$

We adding a new variable say $\lambda^i = [\lambda_0^i, \lambda_1^i, \lambda_2^i, \lambda_3^i]$ to the solution Y_i, $for\ i = 0,1, \dots, 9$of the problem which minimizes the error as follows:

Since $Y_{0\lambda^0} = 1\lambda_1^0 + x\lambda_2^0 + \lambda_3^0 x^2 + 1.0747 \times 10^{-12} \lambda_4^0 x^3$

$\mathrm{Y}_0' = 1\lambda_2^0 + 2\lambda_3^0 t + 3.2241 \times 10^{-12} \lambda_4^0 x^2$

$\mathrm{Y}_0'' = 2\lambda_3^0 + 6.4482 \times 10^{-12} \lambda_4^0 x$

And $(Y_{0\lambda^0})'' + (Y_{0\lambda^0})' + Y_{0\lambda^0} = x^2 + 3x + 4$

$\therefore\ 1\lambda_1^0 + (1 + x)\lambda_2^0 + (2 + 2x + x^2)\lambda_3^0 + (6.4482 \times 10^{-12} x + 3.2241 \times 10^{-12} x^2 + 1.0747 \times 10^{-12} x^3)\lambda_4^0 = x^2 + 3x + 4$

Let $w_1^0 = 1, w_2^0 = 1 + x, w_2^0 = 2 + 2x + x^2$,

$w_3^0 = 6.4482 \times 10^{-12} x + 3.2241 \times 10^{-12} x^2 + 1.0747 \times 10^{-12} x^3$

And the solution of the linear system w.r.t λ^0, we have

$\lambda_1^0 = 1, \quad \lambda_2^0 = 1, \quad \lambda_3^0 = 1, \quad \lambda_4^0 = 0$

And hence $Y_{0\lambda^0} = x^2 + x + 1$ which represent the exact solution on$[0,1]$

This table shows the cubic b-spline solutions with new extended b-spline and exact solution.

$[x_i, x_{i+1})$	B-spline Y_i	Extended $Y_{i\lambda^i}$	Y_{exact}
$[0, 0.1)$	$Y_0 = 1 + x + x^2 + 1.0747 \times 10^{-12}x^3$	$Y_{0\lambda^0} = x^2 + x + 1$	$Y_0 = x^2 + x + 1$
$[0.1, 0.2)$	$Y_1 = 1 + x + x^2 + 1.0974 \times 10^{-12}x^3$	$Y_{1\lambda^1} = x^2 + x + 1$	$Y_1 = x^2 + x + 1$
$[0.2, 0.3)$	$Y_2 = 1 + x + x^2 - 3.3307 \times 10^{-12}x^3$	$Y_{2\lambda^2} = x^2 + x + 1$	$Y_2 = x^2 + x + 1$
$[0.3, 0.4)$	$Y_3 = 1 + x + x^2 + 1.8874 \times 10^{-12}x^3$	$Y_{3\lambda^3} = x^2 + x + 1$	$Y_3 = x^2 + x + 1$
$[0.4, 0.5)$	$Y_4 = 1 + x + x^2 + 2.5905 \times 10^{-13}x^3$	$Y_{4\lambda^4} = x^2 + x + 1$	$Y_4 = x^2 + x + 1$
$[0.5, 0.6)$	$Y_5 = 1 + x + x^2 + 2.4376 \times 10^{-13}x^3$	$Y_{5\lambda^5} = x^2 + x + 1$	$Y_5 = x^2 + x + 1$
$[0.6, 0.7)$	$Y_6 = 1 + x + x^2 - 1.8198 \times 10^{-12}x^3$	$Y_{6\lambda^6} = x^2 + x + 1$	$Y_6 = x^2 + x + 1$
$[0.7, 0.8)$	$Y_7 = 1 + x + x^2 + 1.0550 \times 10^{-12}x^3$	$Y_{7\lambda^7} = x^2 + x + 1$	$Y_7 = x^2 + x + 1$
$[0.8, 0.9)$	$Y_8 = 1 + x + x^2$	$Y_{8\lambda^8} = x^2 + x + 1$	$Y_8 = x^2 + x + 1$
$[0.9, 1)$	$Y_9 = 1 + x + x^2$	$Y_{9\lambda^9} = x^2 + x + 1$	$Y_9 = x^2 + x + 1$

Table (2): cubic B-spline solution with extended B-spline and exact solution

Example:

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 5 + \frac{16}{x} - \frac{1}{x^2} \qquad on\ [0,1]$$

$$with\ y(1) = 8,\ \ y(2) = 19 \quad , \ for\ \ n = 10$$

The exact solution $y = x^2 + 8x - 1$

This table shows the cubic b-spline solutions with new extended b-spline and exact solution.

x_i	B-spline Y_i	Extended $Y_{i\lambda^i}$	Y_{exact}
[0,0.1)	$Y_0 = -1 + 8x + x^2 + 1.6512 \times 10^{-11}x^3$	$Y_{0\lambda^0} = x^2 + 8x - 1 = 1$	$Y_0 = x^2 + 8x - 1 = 1$
[0.1,0.2)	$Y_1 = -1 + 8x + x^2 - 3.1329 \times 10^{-11}x^3$	$Y_{1\lambda^1} = x^2 + 8x - 1 = -1$	$Y_1 = x^2 + 8x - 1 = -1$
[0.2,0.3)	$Y_2 = -1 + 8x + x^2 + 3.7165 \times 10^{-11}x^3$	$Y_{2\lambda^2} = x^2 + 8x - 1 = 0.6400$	$Y_2 = x^2 + 8x - 1 = 0.6400$
[0.3,0.4)	$Y_3 = -1 + 8x + x^2 - 2.6977 \times 10^{-11}x^3$	$Y_{3\lambda^3} = x^2 + 8x - 1 = 1.4900$	$Y_3 = x^2 + 8x - 1 = 1.4900$
[0.4,0.5)	$Y_4 = -1 + 8x + x^2 + 1.7053 \times 10^{-12}x^3$	$Y_{4\lambda^4} = x^2 + 8x - 1 = 2.3600$	$Y_4 = x^2 + 8x - 1 = 2.3600$
[0.5,0.6)	$Y_5 = -1 + 8x + x^2 + 1.5296 \times 10^{-11}x^3$	$Y_{5\lambda^5} = x^2 + 8x - 1 = 3.2500$	$Y_5 = x^2 + 8x - 1 = 3.2500$
[0.6,0.7)	$Y_6 = -1 + 8x + x^2 - 2.1672 \times 10^{-12}x^3$	$Y_{6\lambda^6} = x^2 + 8x - 1 = 4.1600$	$Y_6 = x^2 + 8x - 1 = 4.16600$
[0.7,0.8)	$Y_7 = -1 + 8x + x^2 - 1.2407 \times 10^{-11}x^3$	$Y_{7\lambda^7} = x^2 + 8x - 1 = 5.0900$	$Y_7 = x^2 + 8x - 1 = 5.0900$
[0.8,0.9)	$Y_8 = -1 + 8x + x^2 + 6.5646 \times 10^{-12}x^3$	$Y_{8\lambda^8} = x^2 + 8x - 1 = 6.0400$	$Y_8 = x^2 + 8x - 1 = 6.0400$
[0.9,1)	$Y_9 = -1 + 8x + x^2$	$Y_{9\lambda^9} = x^2 + 8x - 1 = 7.0100$	$Y_9 = x^2 + 8x - 1 = 7.0100$

Table (3): cubic B-spline solution with extended B-spline and exact solution.

Conclusion

In this paper, the extended of cubic B-spline with continuous least square approximation has been used to solve the boundary value problem of second order ordinary differential equation by adding a new vector $\lambda \in R^{n+1}$. The numerical results showed that the extended cubic B-spline approximations, the exact solution of boundary value problems considered very well.

References

[1] Abd Hamid. N. N. B., (2010), " Splines for linear two-point boundary value problems", University Sains Malaysia, ph. D. Thesis.

[2] Abd Hamid. N. N., Majid. A. A. & Ismail. A. I. M., (2011), "Extended Cubic B-spline Method for Linear Two-Point Boundary Value Problems", University Sains Malaysia 11800 USM, pp. 1285–1290, Vol. 4, No. 8.

[3] Abd Hamid. N. N, Majid. A. A. & Ismail. A. I. M., (2012), "Bicubic B-spline interpolation method for two-dimensional Laplace's equations", Prosiding Kolokium Kebangsaan Pasca Siswazah Sains dan Matematik, No. 2063-59-9, pp. 978-983, Vol. 8.

[4] Chang. J., Yang. Q. & Zhao. L., (2011), "Comparison of B-spline Method and Finite Difference Method to Solve BVP of Linear ODEs", Journal of Computers, Vol. 6, No. 10.

[5] Johnson. R. W., (2003), "Progress on the development of B-spline Collocation for the solution of differential model equations: a novel algorithm for adaptive knot insertion", Idaho National Engineering and Environmental Laboratory, USA, 11 pages, Vol. 5, No. 7.

[6] Kaur. H., (2012), "Numerical Solutions of Some Differential Equations Using B-Spline Collocation Method", Thapar University India, ph. D. Thesis.

[7] Majid. A. A., Abbas. M., Ismail. I. A. and Rashid. A., "Numerical Method Using Cubic Trigonometric B-Spline Technique for non-classical Diffusion Problems", Hindawi Publishing, Vol. 2014, Article ID 849682, 11 pages.

[8] Rashidinia. J. and Sharifi. S., (2012), " Survey of B-spline functions to approximate the solution of mathematical problems", Rashidinia and Sharifi Mathematical Sciences.

[9] Yah Ru. J. G., (2013), " B-splines for initial and boundary value problems", University Sains Malaysia, ph. D. Thesis.

توسيع مسار B-Spline بأستخدام التقريب المربع الأقل استمرارية

د. سعد شاكر محمود و نور قاسم علي

الجامعة المستنصرية / كلية التربية / قسم الرياضيات

المستخلص

قمنا بحل مسألة الشروط الحدودية للمعادلة التفاضلية من الرتبة الثانية بأستخدام توسيع جديد لطريقة B-Spline وذلك بأضافة باراميتر جديد $\lambda \in R^{n+1}$. وهذه الطريقة تعطي نتائج افضل من طريقة B-Spline.