Linear Formula Estimators For The Reliability Of Transmuted Pareto Distribution

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Abstract

In this paper, been studying an estimation of reliability for Transmuted Pareto (TP) distribution, by methods (Least Square, Weighted Least Square, Regression, White and New White) . We used simulation to generate data for three experimental cases (E_1, E_2, E_3) of default values for the real parameters with the sample size $(n = 10, 30, 70,100)$ and sample replicated$(N = 1000)$, values for reliability times were taken $t_{(i)} = 0.1,0.2, \dots \le x_{(n)}$.

The comparison operation of the results obtained was conducted using mean square error (MSE),the results showed the following :

In (E_1) was found that White and Regression methods are the best methods of estimating reliability, but in (E_2, E_3) the White method was the best.

Keywords: Transmuted Pareto Distribution, Estimation methods, Simulation, Mean square error.

مقدرات الصيغ الخطية لمعولية توزيع باريتو المحول

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الخلاصة

في هذا البحث ، تم دراسة تقدير معولية توزيع باريتو المحول، من خلال الطرائق (المربعات الصغرى ، المربعات الصغرى الموزونة ، الانحدار ، وايت و وايت الجديد). استخدمنا اسلوب المحاكاة لإنشاء البيانات لثلاث حالات تجريبية (E_1, E_2, E_3) للقيم الافتراضية للمعلمات الحقيقية مع حجم العينة $(n = 10,30,70,100)$ و العينة المكررة (N = 1000) ، تم أخذ قيم أوقات المعولية $t_{(i)} = 0.1,0.2, \dots \le x_{(n)}$.

أجريت عملية مقارنة للنتائج التي تم الحصول عليها باستخدام متوسط مربع الخطأ (MSE) ، اظهرت النتائج ما يلي :

في (E_1) وجدت ان طريقتي وايت والانحدار هما الافضل من الطرائق الاخرى في تقدير المعولية ، اما في (E_3, E_2) فان طريقة وايت كانت هي الافضل .

1. Introduction

Vilfredo Pareto (1897) have been suggested The Pareto distribution, with Probability density function (pdf) and Cumulative distribution function (cdf), respectively as [4].

$$g(x;\beta,x_0) = \frac{\beta x_0^{\beta}}{x^{\beta+1}} \ ,$$

$$G(x;\beta,x_0) = 1 - \left(\frac{x_0}{x}\right)^{\beta},$$

Where $\beta > 0$ is the shape parameter, X is a value of random variable with $x > x_0$ and $0 < x_0 < x_{(1)}$.

Faton and Llukan (2014) proposed Transmuted Pareto (TP) distribution, which has (pdf) and (cdf), respectively as[2]:

$$f_{TP}(x; x_0, \beta, \theta) = \frac{\beta x_0^{\beta}}{x^{\beta+1}}\left[1 - \theta + 2\theta\left(\frac{x_0}{x}\right)^{\beta}\right] \quad \dots (1)$$

$$F_{TP}(x; x_0, \beta, \theta) = \left[1 - \left(\frac{x_0}{x}\right)^{\beta}\right]\left[1 + \theta\left(\frac{x_0}{x}\right)^{\beta}\right] \quad \dots (2)$$

Where $|\theta| \leq 1$ is transmutation parameter.
The reliability function is given by

$$R(t) = \left(\frac{x_0}{t}\right)^{\beta}\left(\theta\left(\frac{x_0}{t}\right)^{\beta} - \theta + 1\right) \quad \dots (3)$$

The hazard ratio function

$$h(x) = \frac{\beta\left(1 - \theta + 2\theta\left(\frac{x_0}{x}\right)^{\beta}\right)}{x\left(1 - \theta + \theta\left(\frac{x_0}{x}\right)^{\beta}\right)} \quad \dots (4)$$

Quantial function of the (TP) distribution is

$$x(p) = x_0\left[\frac{\theta - 1 + \sqrt{(1-\theta)^2 - 4\theta(p-1)}}{2\theta}\right]^{\frac{-1}{\beta}} \quad \dots (5)$$

The aim of this study is to find the best estimate of the parameters and reliability function of the Transmuted Pareto distribution. To achieve the aim of the study, this research was divided into four sections. The first section included an introduction in which the Transmuted Pareto distribution was included, and in the second section five methods were derived to find the estimation of each of the shape parameter (β) and transmutation parameter (θ) and the reliability function, the third section the experiments and the results was presented, and the fourth section included the conclusions of the simulation experiment that show the best methods for estimating both the parameters and the reliability function of the distribution.

2. Estimation Methods

In this section, we study different estimates of the unknown parameters (β, θ) and reliability function $R(t)$ of the Transmuted Pareto (TP) distribution.

2.1 Estimate Initial Value for Estimators:

Makki and Jaafer (2019) suggested the following technique to find the initial values for estimating the parameters through any distribution [6] .
the median of (TP) distribution is[2] .

$$x_{med} = \frac{x_0}{\left[\frac{\theta - 1 + \sqrt{1+\theta^2}}{2\theta}\right]^{\frac{1}{\beta}}}$$

$$\left[\frac{\theta - 1 + \sqrt{1+\theta^2}}{2\theta}\right]^{\frac{1}{\beta}} = \frac{x_0}{x_{med}} \quad \dots (6)$$

Taking the natural logarithm for equation (6), will be as:

$$\frac{1}{\beta} ln\left[\frac{\theta - 1 + \sqrt{1+\theta^2}}{2\theta}\right] = ln\left(\frac{x_0}{x_{med}}\right)$$

$$\hat{\beta}_0 = \frac{ln\left[\frac{\theta - 1 + \sqrt{1+\theta^2}}{2\theta}\right]}{ln\left(\frac{x_0}{x_{med}}\right)} \qquad ...(7)$$

From equation (6)

$$\left(\frac{x_0}{x_{med}}\right)^{\beta} = \frac{\theta - 1 + \sqrt{1+\theta^2}}{2\theta}$$

$$\hat{\theta}_0 = \frac{\theta - 1 + \sqrt{1+\theta^2}}{2\left(\frac{x_0}{x_{med}}\right)^{\beta}} \qquad ...(8)$$

We can use $\hat{\beta}_0$ and $\hat{\theta}_0$ in equations (7) and (8) as Initial values to another parameter estimator formula, x_{med} can be obtained it from the generating sample.

2.2 Least Squares Method (LS):

This method aims to reduce the sum of squares of errors to the minimum after converting the quantial function into a formula similar to the linear regression model [3]. By using equation (5), getting

$$\frac{x(p)}{x_0} = \left[\frac{\theta - 1 + \sqrt{(1-\theta)^2 - 4\theta(p-1)}}{2\theta}\right]^{\frac{-1}{\beta}} \qquad ...(9)$$

Taking the natural logarithm for equation (9), will be as:

$$ln\left(\frac{x(p)}{x_0}\right) = \frac{-1}{\beta}\, ln\left[\frac{\theta - 1 + \sqrt{(1-\theta)^2 - 4\theta(p-1)}}{2\theta}\right] \qquad ...(10)$$

$$\beta\, ln\left(\frac{x(p)}{x_0}\right) = ln(2\theta) - ln\left[\theta - 1 + \sqrt{(1-\theta)^2 - 4\theta(p-1)}\right]$$

$$ln\left[\theta - 1 + \sqrt{(1-\theta)^2 - 4\theta(p-1)}\right] = ln(2\theta) + \beta\left(-\,ln\left(\frac{x(p)}{x_0}\right)\right) \qquad ...(11)$$

$$z_i = a + b\phi_i + \epsilon \qquad ...(12)$$

Comparing equation (11) with equation (12), will be as:

$$z_i = ln\left[\theta - 1 + \sqrt{(1-\theta)^2 - 4\theta(p-1)}\right], b = \beta, \qquad ...(13)$$

$$a = ln(2\theta) \;\Rightarrow \theta = 0.5\, e^{a} \qquad ...(14)$$

$$\phi_i = -\,ln\left(\frac{x(p)}{x_0}\right) \qquad ...(15)$$

From equation (12), getting

$$\epsilon = z_i - a - b\phi_i \qquad ...(16)$$

Squaring equation (16) and taking the sum of both sides, will be as:

$$\sum_{i=1}^{n}\epsilon^2 = \sum_{i=1}^{n}(z_i - a - b\phi_i)^2 \qquad ...(17)$$

Let $\delta = \sum_{i=1}^{n}\epsilon^2$

$$\delta(a,b) = \sum_{i=1}^{n}(z_i - a - b\phi_i)^2 \qquad ...(18)$$

The partial derivative equation (18) with respect to a and equal to zero, and the partial derivative equation (18) with respect to b and equal to zero, will be as:

$$a = \frac{\sum_{i=1}^{n} z_i - b\sum_{i=1}^{n} \phi_i}{n} \qquad \dots(19)$$

$$b = \frac{\sum_{i=1}^{n} z_i \sum_{i=1}^{n} \phi_i - n\sum_{i=1}^{n} z_i\phi_i}{(\sum_{i=1}^{n} \phi_i)^2 - n\sum_{i=1}^{n} \phi_i^2} \qquad \dots(20)$$

Compensated equation (13) in equation (20), will be as:

$$\hat{b} = \hat{\beta}_{LS} = \frac{\begin{pmatrix} \sum_{i=1}^{n} ln\left[\theta_0 - 1 + \sqrt{(1-\theta_0)^2 - 4\theta_0(p-1)}\right]\sum_{i=1}^{n} \phi_i \\ -n\sum_{i=1}^{n} ln\left[\theta_0 - 1 + \sqrt{(1-\theta_0)^2 - 4\theta_0(p-1)}\right]\phi_i \end{pmatrix}}{(\sum_{i=1}^{n} \phi_i)^2 - n\sum_{i=1}^{n} \phi_i^2} \qquad \dots(21)$$

From equation (19) and (20), will be as:

$$\hat{a} = \frac{\sum_{i=1}^{n} z_i - \hat{b}\sum_{i=1}^{n} \phi_i}{n} \qquad \dots(22)$$

By substituting equation (22) into equation (14), will be as:

$$\hat{\theta}_{LS} = 0.5\, e^{\hat{a}} \qquad \dots(23)$$

Approximate reliability can be estimated by substituting two equations (21) and (23) into equation (3) as follows:

$$\hat{R}(t)_{LS} = \left(\frac{x_0}{t}\right)^{\hat{\beta}_{LS}} \left(\hat{\theta}_{LS}\left(\frac{x_0}{t}\right)^{\hat{\beta}_{LS}} - \hat{\theta}_{LS} + 1\right) \qquad \dots(24)$$

2.3 Weighted Least Squares Method (WLS) :

This method depends on equation (17) by multiplying it $\left(\frac{1}{z_i}\right)$, squaring it and taking the sum of both sides, to obtain: [7].

$$\sum_{i=1}^{n} \left(\frac{\epsilon}{z_i}\right)^2 = \sum_{i=1}^{n} \left(1 - \frac{1}{z_i}a - \frac{\phi_i}{z_i}b\right)^2 \qquad \dots(25)$$

Let $\delta = \sum_{i=1}^{n} \left(\frac{\epsilon}{z_i}\right)^2,\ \sigma_i = \frac{1}{z_i}\ \ ,\varphi_i = \frac{\phi_i}{z_i}$ ... (26)

Substituted equation (26) into equation (25), will be as:

$$\delta = \sum_{i=1}^{n} (1 - \sigma_i a - \varphi_i b)^2 \qquad \dots(27)$$

Now, driving equation (27) (w.r.t) a and b ,and equaling to zero, getting two equations are respectively:

$$\frac{\partial\, \delta}{\partial a} = 2\sum_{i=1}^{n} (1 - a\sigma_i - b\varphi_i)\,(-\sigma_i) = 0$$

$$a\sum_{i=1}^{n} \sigma_i^2 = \sum_{i=1}^{n} \sigma_i - b\sum_{i=1}^{n} \sigma_i\varphi_i$$

$$a = \frac{\sum_{i=1}^{n} \sigma_i - b\sum_{i=1}^{n} \sigma_i\varphi_i}{\sum_{i=1}^{n} \sigma_i^2} \qquad \dots(28)$$

Now,

$$\frac{\partial\, \delta}{\partial b} = 2\sum_{i=1}^{n} (1 - a\sigma_i - b\varphi_i)\,(-\varphi_i) = 0$$

$$a\sum_{i=1}^{n} \sigma_i\varphi_i = \sum_{i=1}^{n} \varphi_i - b\sum_{i=1}^{n} \varphi_i^2$$

$$a = \frac{\sum_{i=1}^{n} \varphi_i - b\sum_{i=1}^{n} \varphi_i^2}{\sum_{i=1}^{n} \sigma_i\varphi_i} \qquad \dots(29)$$

By equality equation (28) with equation (29), will be as:

$$b\sum_{i=1}^{n}\varphi_i^2\sum_{i=1}^{n}\sigma_i^2 - b\left(\sum_{i=1}^{n}\sigma_i\varphi_i\right)^2 = \sum_{i=1}^{n}\varphi_i\sum_{i=1}^{n}\sigma_i^2 - \sum_{i=1}^{n}\sigma_i\sum_{i=1}^{n}\varphi_i\,\sigma_i$$

Since $b=\beta$,then

$$\hat{b} = \hat{\beta}_{WLS} = \frac{\sum_{i=1}^{n}\varphi_i\sum_{i=1}^{n}\sigma_i^2 - \sum_{i=1}^{n}\sigma_i\sum_{i=1}^{n}\sigma_i\varphi_i}{\sum_{i=1}^{n}\varphi_i^2\sum_{i=1}^{n}\sigma_i^2 - (\sum_{i=1}^{n}\sigma_i\varphi_i)^2} \quad \ldots(30)$$

Now, substituted equation (30) in equation (29), will be as:

$$\hat{a} = \frac{\sum_{i=1}^{n}\sigma_i - \hat{\beta}_{WLS}\sum_{i=1}^{n}\sigma_i\varphi_i}{\sum_{i=1}^{n}\sigma_i^2} \quad \ldots(31)$$

By substituting equation (31) into equation (14), will be as:

$$\hat{\theta}_{WLS} = 0.5\,e^{\hat{a}} \quad \ldots(32)$$

Approximate reliability can be estimated by substituting two equations (30) and (32) into equation (3) as follows:

$$\hat{R}(t)_{WLS} = \left(\frac{x_0}{t}\right)^{\hat{\beta}_{WLS}}\left(\hat{\theta}_{WLS}\left(\frac{x_0}{t}\right)^{\hat{\beta}_{WLS}} - \hat{\theta}_{WLS} + 1\right) \quad (33)$$

2.4 Regression Estimation Method (Re):

This method depends on converting the cumulative distribution function formula into a formula similar for the linear regression equation [1] .

From equation (2), will be as:

$$\frac{F(x)}{1-\left(\frac{x_0}{x}\right)^{\beta}} = 1 + \theta\left(\frac{x_0}{x}\right)^{\beta}$$

$$\frac{F(x)}{1-\left(\frac{x_0}{x}\right)^{\beta}} - 1 = \theta\left(\frac{x_0}{x}\right)^{\beta} \quad \ldots(34)$$

Taking the natural logarithm for the equation (34), so will be as:

$$ln\left(\frac{F(x_i)}{1-\left(\frac{x_0}{x_i}\right)^{\beta}} - 1\right) = ln\theta + \beta ln\left(\frac{x_0}{x_i}\right) \quad \ldots(35)$$

$$k_i = s + l\zeta_i + \epsilon \quad \ldots(36)$$

Comparing equation (36) with the equation (35), will be as:

$$k_i = ln\left(\frac{F(x_i)}{1-\left(\frac{x_0}{x_i}\right)^{\beta}} - 1\right), \quad \zeta_i = ln\left(\frac{x_0}{x_i}\right)$$

$$l = \beta \quad , s = ln\,\theta \;\Rightarrow \theta = e^{s}$$

$$\bar{\zeta} = \frac{\sum_{i=1}^{n} ln\left(\frac{x_0}{x_i}\right)}{n}$$

$$\bar{k} = \frac{\sum_{i=1}^{n} ln\left(\frac{F(x_i)}{1-\left(\frac{x_0}{x_i}\right)^{\beta}} - 1\right)}{n}$$

$$\hat{\beta}_{Re} = \hat{l} = \frac{\sum_{i=1}^{n}(\zeta_i - \bar{\zeta})\,k_i}{\sum_{i=1}^{n}(\zeta_i - \bar{\zeta})^2} \quad \ldots(37)$$

From equation (36), will be as:

$$ns = \sum_{i=1}^{n} k_i - l \sum_{i=1}^{n} \zeta_i$$

$$s = \bar{k} - \hat{\beta}_{Re} \bar{\zeta} \qquad \dots (38)$$

$$\hat{\theta}_{Re} = e^{s} \qquad \dots (39)$$

Substituting in equation (41) into equation (40), will be as:

$$\hat{\theta}_{Re} = e^{\bar{k} - \hat{\beta}_{Re} \bar{\zeta}} \qquad \dots (40)$$

Approximate reliability can be estimated by substituting two equations (37) and (40) into equation (3) as follows:

$$\hat{R}(t)_{Re} = \left(\frac{x_0}{t}\right)^{\hat{\beta}_{Re}} \left(\hat{\theta}_{Re} \left(\frac{x_0}{t}\right)^{\hat{\beta}_{Re}} - \hat{\theta}_{Re} + 1\right) \qquad \dots (41)$$

2.5 White Method (W):

This method depends on the reliability function of the distribution whose parameters are to be estimated, by converting the formula of the reliability function into a formula similar to the formula for the linear regression equation [5].

From equation (3), will be as:

$$R(x) = \left(\frac{x_0}{x}\right)^{\beta} \theta \left(\left(\frac{x_0}{x}\right)^{\beta} - 1 + \frac{1}{\theta}\right) \qquad \dots (42)$$

Taking the natural logarithm for equation (42), so will be as:

$$ln\big(R(x_i)\big) = \beta \, ln\left(\frac{x_0}{x_i}\right) + ln\,\theta + ln\left(\left(\frac{x_0}{x_i}\right)^{\beta} - 1 + \frac{1}{\theta}\right)$$

$$ln\big(R(x_i)\big) - ln\left(\left(\frac{x_0}{x_i}\right)^{\beta} - 1 + \frac{1}{\theta}\right) = \beta \, ln\left(\frac{x_0}{x_i}\right) + ln\,\theta \qquad \dots (43)$$

Comparing equation (43) with equation (36), will be as:

$$k_i = ln\big(R(x_i)\big) - ln\left(\left(\frac{x_0}{x_i}\right)^{\beta} - 1 + \frac{1}{\theta}\right)$$

$$s = ln\,\theta \Rightarrow \theta = e^{s} \quad , \; l = \beta \quad , \zeta_i = ln\left(\frac{x_0}{x_i}\right)$$

$$\bar{\zeta} = \frac{\sum_{i=1}^{n} ln\left(\frac{x_0}{x_i}\right)}{n} \;\; ; \bar{k} = \frac{\sum_{i=1}^{n} ln\big(R(x_i)\big) - ln\left(\left(\frac{x_0}{x_i}\right)^{\beta} - 1 + \frac{1}{\theta}\right)}{n}$$

$$\hat{\beta}_W = \hat{l} = \frac{\sum_{i=1}^{n} (\zeta_i - \bar{\zeta})(k_i - \bar{k})}{\sum_{i=1}^{n} (\zeta_i - \bar{\zeta})^2} \qquad \dots (44)$$

From equation (36), will be as:

$$s = \bar{k} - \hat{\beta}_W \bar{\zeta} \qquad \dots (45)$$

$$\hat{\theta}_W = e^{s} \qquad \dots (46)$$

Substituting in equation (46) into equation (45), will be as:

$$\hat{\theta}_W = e^{\bar{k} - \hat{\beta}_W \bar{\zeta}} \qquad \dots (47)$$

Approximate reliability can be estimated by substituting two equations (44) and (٤7) into equation (3) as follows:

$$\hat{R}(t)_W = \left(\frac{x_0}{t}\right)^{\hat{\beta}_W} \left(\hat{\theta}_W \left(\frac{x_0}{t}\right)^{\hat{\beta}_W} - \hat{\theta}_W + 1\right) \qquad \dots (48)$$

2.6 New White Method (NW):

This method depends on converting the hazard function formula into a formula similar to the formula for the linear regression equation[٥].

From equation (4), will be as:

$$h(x)=\frac{\frac{\beta}{x}(1-\theta)+\frac{2\beta}{x}\theta\left(\frac{x_0}{x}\right)^{\beta}}{\left(1-\theta+\theta\left(\frac{x_0}{x}\right)^{\beta}\right)}$$

$$h(x)=\frac{\frac{\beta}{x}(1-\theta)}{\left(1-\theta+\theta\left(\frac{x_0}{x}\right)^{\beta}\right)}+\frac{\frac{2\beta}{x}}{\left(1-\theta+\theta\left(\frac{x_0}{x}\right)^{\beta}\right)}\theta\left(\frac{x_0}{x}\right)^{\beta}$$

$$h(x)-\frac{\frac{\beta}{x}(1-\theta)}{\left(1-\theta+\theta\left(\frac{x_0}{x}\right)^{\beta}\right)}=\frac{\frac{2\beta}{x}}{\left(1-\theta+\theta\left(\frac{x_0}{x}\right)^{\beta}\right)}\theta\left(\frac{x_0}{x}\right)^{\beta} \quad \dots(49)$$

Taking the natural logarithm for equation (49), so will be as:

$$ln\left(h(x_i)-\frac{\frac{\beta}{x_i}(1-\theta)}{\left(1-\theta+\theta\left(\frac{x_0}{x_i}\right)^{\beta}\right)}\right)=ln\left(\frac{\frac{2\beta}{x_i}}{\left(1-\theta+\theta\left(\frac{x_0}{x_i}\right)^{\beta}\right)}\right)+ln\theta+\beta\, ln\left(\frac{x_0}{x_i}\right)$$

$$ln\left(h(x_i)-\frac{\frac{\beta}{x_i}(1-\theta)}{\left(1-\theta+\theta\left(\frac{x_0}{x_i}\right)^{\beta}\right)}\right)-ln\left(\frac{\frac{2\beta}{x_i}}{\left(1-\theta+\theta\left(\frac{x_0}{x_i}\right)^{\beta}\right)}\right)=\beta\, ln\left(\frac{x_0}{x_i}\right)+ln\theta \quad \dots(50)$$

Comparing equation (50) with equation (36), will be as:

$$k=ln\left(h(x_i)-\frac{\frac{\beta}{x_i}(1-\theta)}{\left(1-\theta+\theta\left(\frac{x_0}{x_i}\right)^{\beta}\right)}\right)-ln\left(\frac{\frac{2\beta}{x_i}}{\left(1-\theta+\theta\left(\frac{x_0}{x_i}\right)^{\beta}\right)}\right)$$

$$s=ln\,\theta \Rightarrow \theta=e^{s} \quad , \quad l=\beta \quad ,\zeta_i=ln\left(\frac{x_0}{x_i}\right)$$

$$\bar{\zeta}=\frac{\sum_{i=1}^{n} ln\left(\frac{x_0}{x_i}\right)}{n}$$

$$\bar{k}=\frac{\sum_{i=1}^{n}\left(ln\left(h(x_i)-\frac{\frac{\beta}{x_i}(1-\theta)}{\left(1-\theta+\theta\left(\frac{x_0}{x_i}\right)^{\beta}\right)}\right)-ln\left(\frac{\frac{2\beta}{x_i}}{\left(1-\theta+\theta\left(\frac{x_0}{x_i}\right)^{\beta}\right)}\right)\right)}{n}$$

$$\hat{\beta}_{NW}=\hat{l}=\frac{\sum_{i=1}^{n}(\zeta_i-\bar{\zeta})(k_i-\bar{k})}{\sum_{i=1}^{n}(\zeta_i-\bar{\zeta})^2} \quad \dots(51)$$

From equation (36), will be as:

$$s=\bar{k}-\hat{\beta}_{NW}\bar{\zeta} \quad \dots(52)$$

$$\hat{\theta}_{NW}=e^{s} \quad \dots(53)$$

Substituting in equation (53) into equation (52), will be as:

$$\hat{\theta}_{NW}=e^{\bar{k}-\hat{\beta}_{NW}\bar{\zeta}} \quad \dots(54)$$

Approximate reliability can be estimated by substituting two equations (51) and (54) into equation (3) as follows:

$$\hat{R}(t)_{NW}=\left(\frac{x_0}{t}\right)^{\hat{\beta}_{NW}}\left(\hat{\theta}_{NW}\left(\frac{x_0}{t}\right)^{\hat{\beta}_{NW}}-\hat{\theta}_{NW}+1\right) \quad \dots(55)$$

3. Experiments and Results:

In this section, we present simulation steps in terms of selecting sample sizes, real values of parameters and life time values that were used to estimate reliability:

1. The sample size (n=10, 30, 70, 100), and sample replicated (N=1000).
2. Several values of the shape and transmutation parameters (β, θ), as show in table (1) below:

Table 1-The default value for parameters

Cases		E_1	E_2	E_3
parameters	β	**2.5**	**1.5**	**1**
	θ	**0.5**	**0.1**	**0.5**

3. In all cases, (E_1, E_2, E_3), the life time for estimating reliability chosen $x_0 \leq t \leq x_{(n)}$, such that, $t_{(i)} = 0.1, 0.2, \dots \leq x_{(n)}$.
4. At this stage, random data is generated of (TP) distribution by equation (4) and using MATLAB language version R2012b.
5. At this stage finding the value of parameter and reliability estimated according to the equations(21),(23),(24),(30),(32),(33),(37),(40),(41),(44),(47),(48),(51),(54)and (55) .
6. Finally, comparison between the estimators is done by $MSE(\hat{\alpha}) = \frac{\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2}{N}$, where$(\hat{\alpha})$ is an estimator for parameter (α).

Tables (2,3 and 4) , bellow , shows the empirical value for estimation of parameters and reliability function

Table 2- Estimated values for (R) and (β, θ) using $(E_1 : R = 0.55200)$

Methods	n	Mean Estimated Values			MSE		
		$\hat{\beta}$	$\hat{\theta}$	$\hat{R}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{R}$
LS	10	1.02051	0.81643	0.64443	2.18886	0.10013	0.01708
WLS		0.66083	0.75033	0.72905	3.38252	0.06266	0.06268
Re		1.02945	0.99426	0.44894	2.16249	0.24429	0.00798
W		1.53429	0.67687	0.59611	0.93258	0.03128	0.00389
NW		2.78453	0.54171	0.55615	2.27156	0.02496	0.00208
LS	30	1.64520	0.67310	0.41501	0.73067	0.02996	0.00207
WLS		1.04599	0.58906	0.50885	2.11412	0.00793	0.02615
Re		1.61016	0.65232	0.59193	0.79180	0.02320	0.00318
W		2.75337	0.47834	0.54394	0.06420	0.00046	0.00013
NW		2.61676	0.51085	0.23447	0.56680	0.00498	0.00050
LS	70	3.04671	0.56040	0.53070	0.29889	0.00364	0.00090
WLS		2.17014	0.52186	0.56599	0.10880	0.00047	0.00039
Re		2.48366	0.50158	0.16792	0.00026	0.00251×10^{-3}	0.00025×10^{-3}
W		2.37309	0.51303	0.17050	0.01610	0.00016	0.16750×10^{-4}
NW		2.55778	0.50425	0.28901	0.22121	0.00204	0.00028
LS	100	2.46358	0.44171	0.17086	0.00132	0.00339	0.00002
WLS		2.73977	0.49210	0.28123	0.05749	0.00006	0.00009
Re		2.49814	0.50017	0.28862	0.34484×10^{-5}	0.00320×10^{-5}	0.00054×10^{-5}
W		2.40266	0.50984	0.29167	0.00947	0.00009	0.16020×10^{-4}
NW		2.53076	0.50378	0.28949	0.14653	0.00140	0.00021

Table 3- Estimated values for (R) and (β, θ)using $(E_2: R = 0.12193)$

Methods	n	Mean Estimated Values			MSE		
		$\hat{\beta}$	$\hat{\theta}$	$\hat{R}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{R}$
LS	10	0.80208	0.19311	0.36952	0.48708	0.00867	0.02237
WLS		0.70134	0.18162	0.40969	0.63785	0.00666	0.03699
Re		0.64018	0.23763	0.40034	0.73927	0.01894	0.04185
W		0.93430	0.14772	0.45039	0.32001	0.00227	0.01526
NW		1.67106	0.11119	0.35118	0.74605	0.00169	0.00902
LS	30	1.12324	0.14747	0.26494	0.14194	0.00225	0.00371
WLS		1.01740	0.13948	0.28999	0.23289	0.00155	0.00774
Re		0.97873	0.14095	0.39150	0.27171	0.00167	0.01166
W		1.64811	0.09437	0.24156	0.02193	0.00003	0.00039
NW		1.56782	0.10314	0.34340	0.19659	0.00035	0.00359
LS	70	0.80569	0.11506	0.22425	0.09345	0.00022	0.00162
WLS		1.74228	0.11194	0.23038	0.05870	$0.14263*10^{-3}$	0.00109
Re		1.59669	0.09617	0.11245	0.00935	$0.14651*10^{-4}$	0.00008
W		1.42540	0.10343	0.06293	0.00556	$0.11767*10^{-4}$	0.00003
NW		1.53170	0.10126	0.40509	0.07417	$0.14024*10^{-3}$	0.00181
LS	100	1.46618	0.08593	0.06110	0.00114	0.00019	0.00001
WLS		1.54987	0.09519	0.11612	0.00248	0.00002	0.00002
Re		1.45808	0.10186	0.02441	0.00175	$0.34960*10^{-5}$	$0.37225*10^{-5}$
W		1.51399	0.09940	0.04298	0.19579	0.00035	0.000726
NW		1.51774	0.10107	0.13609	0.05075	0.00009	0.00056

Table 4- Estimated values for (R) and (β, θ) using $(E_3: R = 0.10352)$

Methods	n	Mean Estimated Values			MSE		
		$\hat{\beta}$	$\hat{\theta}$	$\hat{R}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{R}$
LS	10	0.40170	0.83329	0.31611	0.35795	0.11108	0.02641
WLS		0.25491	0.76868	0.47066	0.55514	0.07219	0.10259
Re		0.41175	0.99433	0.25348	0.34603	0.24437	0.01331
W		0.61355	0.67701	0.35027	0.14933	0.03133	0.00854
NW		1.11393	0.54047	0.37668	0.34519	0.02314	0.00699
LS	30	0.65800	0.67324	0.20233	0.11696	0.03001	0.00354
WLS		0.41790	0.58916	0.33727	0.33883	0.00794	0.03962
Re		0.64404	0.65234	0.29748	0.12670	0.02320	0.00634
W		1.10067	0.47846	0.16879	0.01013	0.00046	0.00024
NW		1.04133	0.51231	0.14991	0.08761	0.00517	0.00144
LS	70	1.21864	0.56045	0.14537	0.04780	0.00365	0.00183
WLS		0.86815	0.52193	0.20679	0.01738	0.00048	0.00069
Re		1.06586	0.48534	0.06545	0.00433	0.00021	0.00004
W		0.94907	0.51307	0.02971	0.00259	0.00017	0.00001
NW		1.01876	0.50508	0.13641	0.03475	0.00200	0.00064
LS	100	0.98543	0.44171	0.02980	0.00021	0.00339	$0.16297*10^{-4}$
WLS		1.09591	0.49210	0.06256	0.00919	0.00006	0.00010
Re		0.97138	0.50713	0.00835	0.00081	0.00005	$0.89264*10^{-6}$
W		0.96102	0.50985	0.07433	0.00151	0.00009	$0.17848*10^{-4}$
NW		1.01399	0.50334	0.12544	0.02371	0.00136	0.00041

From the results in Tables 2, 3 and 4, we note the following:

- In the first case(E_1):-

  For estimating the parameter (β), we notice when $(n = 10,30)$ the method (W) is the best, but when $(n = 70,100)$ the method (Re) is the best.

  For estimating the parameter (θ) and the reliability, we notice when $(n = 10)$ the method (NW) is the best, when $(n = 30)$ the method (W) is the best, either when $(n = 70,100)$ the method (Re) is the best.
- In the second and third cases(E_2, E_3):-

For estimating the parameter (β), we notice when ($n = 10,30,70$) the method (W) is the best, either when ($n = 100$) the method (LS) is the best.

For estimating the parameter (θ) and the reliability, we notice when ($n = 10$) the method (NW) is the best, but when ($n = 30,70$) the method (W) is the best, either when ($n = 100$) the method (Re) is the best .

4. Conclusion:

From the results in Tables 2, 3 and 4, we note the following:

- We did not take the negative value of the transmutation parameter (θ) in all estimation methods, because it contains the natural logarithm function in the estimation formulas, so the results will be complex numbers.
- In (E_1) was found that (W) and (Re) methods are the best in most methods in the estimation (β), but in (E_2, E_3) the (W) method was the best.
- In (E_1) was found that (Re) method is the best in most methods in the estimation (θ), but in (E_2, E_3) the (W) method was the best.
- In (E_1) was found that (W) and (Re) methods are the best methods of estimating reliability, but in (E_2, E_3) the (W) method was the best.

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