On Certain Subclasses of Sakaguchi Analytic Functions

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Abstract

In this paper, we study Sakaguchi functions defined in the open unit disk $U\{z \in \mathbb{C}: |z| < 1\}$, certain subclasses of Sakaguchi functions are introduced. Further, some properties like coefficient inequalities, and some properties of the classes $S_0(\alpha)$ and $T_0(\alpha)$, distortion and growth inequalities are presented.

Keywords: Sakaguchi Functions, Analytic Functions.

حول بعض الاصناف الجزئية لدوال ساكاكوشي التحليلية

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الملخص

في هذا البحث قمنا بدراسة دوال ساكاكوشي في قرص الوحدة المفتوح $U\ \{z \in \mathbb{C}: |z| < 1\}$، تم تقديم فئات جزئية معينة من دوال ساكاكوشي. علاوة على ذلك، تم تقديم بعض الخصائص مثل معاملات المتباينات وبعض خصائص الفئات $S_0(\alpha)$ و $T_0(\alpha)$ وتم عرض توسعة وتحديد لمتباينات تلك الفئات.

الكلمات المفتاحية: دوال ساكاكوشي، الدوال التحليلية.

INTRODUCTION

Let B be the class of analytic functions in the open unit disk $U\ \{z \in \mathbb{C}: |z| < 1\}$ defined as follows:

$$f(z) = z + \sum_{m=2}^{\infty} b_m z^m \qquad (1)$$

Where (1) is the Taylor Extended equation. In fact, the function $f(z) \in B$ is said to be in the class $S(\alpha)$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > \alpha \qquad (2)$$

For some $\alpha(0 \le \alpha < \frac{1}{2})$ and for all $z \in U$ where $S(\alpha)$ is class of analytic and univalent function with order α. The class $S(0)$ when $\alpha = 0$ was introduced by Sakaguchi [1] and [6]. Therefore, a function $f(z) \in S(\alpha)$ is called Sakaguchi function of order α which is Univalent function. We also denote by $T(\alpha)$ the subclass of B consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha)$, where $T(\alpha)$ is univalent function with negative coefficient of order α.

For $f(z)$ belonging to $S(\alpha)$ and $T(\alpha)$, Cho, Kwon and Owa [2] and [5] have given the following lemmas:

Lemma 1 Let $f(z) \in B$, such that the following condition is satisfied:

$$\sum_{m=2}^{\infty}\{2(m-1)|b_{2m-2}| + (2m-1-2\alpha)|b_{2m-1}|\} \le 1 - 2\alpha\,, \qquad (3)$$

$\alpha(0 \le \alpha < \frac{1}{2})$. Then $f(z) \in S(\alpha)$.

Lemma 2 Let $f(z) \in B$ such that the following condition is satisfied:

$$\sum_{m=2}^{\infty}\{4(n-1)^2|b_{2m-2}|+(2m-1)(2n-1-2\alpha)|b_{2m-1}|\}\leq 1-2\alpha \quad , \qquad (4)$$

$\alpha(0\leq\alpha<\frac{1}{2})$. Then $f(z)\in T(\alpha)$.

Based on Lemma 1 and Lemma 2, in [2], Cho et al. have introduced the sub-class $S_0(\alpha)$ of $S_s(\alpha)$ which consist of functions that satisfy inequality (3) in Lemma 1, and the subclass $T_0(\alpha)$ of $T_s(\alpha)$ consisting of function which satisfy inequality (4) in Lemma 2. It is easy to see that $T_0(\alpha)\subset S_0(\alpha)$ for $0\leq\alpha<\frac{1}{2}$.

The main results of [2], are given in the following theorems:

Theorem 1 *If* $f(z)\in S_0(\alpha)$ *with* $0\leq\alpha<\frac{1}{2}$, *then*

$$Re\ \frac{f(z)}{z}>\frac{1}{2(1-\alpha)} \quad (z\in E). \qquad (5)$$

Theorem 2 *If* $f(z)\in S_0(\alpha)$ *with* $0\leq\alpha<\frac{1}{2}$, *then*

$$|z|-\frac{(1-2\alpha)}{2}|z|^2-\frac{(1-2\alpha)}{(3-2\alpha)}|z|^3 \leq |f(z)| \leq |z|+\frac{1-2\alpha}{2}|z|^2+\frac{1-2\alpha}{3-2\alpha}|z|^3, \qquad (6)$$

$$1-(1-2\alpha)|z|-\frac{3(1-2\alpha)}{3-2\alpha}|z|^2 \leq |f'(z)| \leq 1+(1-2\alpha)|z|+\frac{3(1-2\alpha)}{3-2\alpha}|z|^2,$$

Theorem 3 *If* $f(z)\in T_0(\alpha)$ *with* $0\leq\alpha<\frac{1}{2}$, *then*

$$|z|-\frac{1-2\alpha}{4}|z|^2-\frac{1-2\alpha}{3(3-2\alpha)}|z|^3 \leq |f(z)| \leq |z|+\frac{1-2\alpha}{4}|z|^2+\frac{1-2\alpha}{3(3-2\alpha)}|z|^3, \qquad (7)$$

$$1-\frac{1-2\alpha}{2}|z|-\frac{1-2\alpha}{3-2\alpha}|z|^2 \leq |f'(z)| \leq 1+\frac{1-2\alpha}{2}|z|+\frac{1-2\alpha}{3-2\alpha}|z|^2,$$

For $z\in U$.

In the rest of this paper, some properties like coefficient inequalities are stated, some properties of the classes $S_0(\alpha)$ and $T_0(\alpha)$ are given. Finally, some distortion and growth inequalities are presented.

COEFFICIENT INEQUALITIES

Applying Carathéodory function

$$p(z)=1+\sum_{m=1}^{\infty}p_m z^m \quad , \qquad (8)$$

In U, we first study the inequalities of coefficients for the function f(z) in $S(\alpha)$ and $T(\alpha)$.

Theorem 4 If $f(z)\in S(\alpha)$, then

$$|a_{2m}|\leq\frac{\prod_{j=1}^{m+1}(j-2\alpha)}{m(m!)} \quad (m\geq 1) \qquad (9)$$

And

$$|a_{2m+1}| \le \frac{\prod_{j=1}^{m}(j-2\alpha)}{m!} \quad (m \ge 1). \qquad (10)$$

Proof We define the function p(z) by

$$p(z) = \frac{1}{1-2\alpha}\left(\frac{2zf'(z)}{f(z)-f(-z)} - 2\alpha\right) = 1 + \sum_{m=1}^{\infty} p_m z^m \qquad (11)$$

For $f(z) \in S(\alpha)$. Then p(z) is a Carathéodory function and satisfies $|p_m| \le 2 \quad (m \ge 1)$.
Since

$$2zf'(z) = \big(f(z) - f(-z)\big)(1-2\alpha)p(z) + 2\alpha,$$

We obtain that

$$a_{2m} = \frac{1-2\alpha}{2m}(p_{2m+1} + a_3 p_{2m-1} + \cdots + a_{2m+1}p_1) \qquad (12)$$

And

$$a_{2m+1} = \frac{1-2\alpha}{2m}(p_{2m} + a_3 p_{2m-2} + \cdots + a_{2m-1}p_2). \qquad (13)$$

Taking $m = 1$, we see that

$$|a_3| \le 1-2\alpha \qquad (14)$$

And

$$|a_2| = \frac{1-2\alpha}{1+|a_3|} \le (1-2\alpha)(2-2\alpha). \qquad (15)$$

Thus, using the mathematical induction, we complete the proof of the theorem.

Theorem 5 If $f(z) \in T(\alpha)$, then

$$|a_{2m}| \le \frac{\prod_{j=1}^{m+1}(j-2\alpha)}{2m^2(m!)} \quad (m \ge 1) \qquad (16)$$

And

$$|a_{2m+1}| \le \frac{\prod_{j=1}^{m}(j-2\alpha)}{(2m+1)(m!)} \quad (m \ge 1). \qquad (17)$$

SOME PROPERTIES OF THE CLASSES $S_0(\alpha)$ AND $T_0(\alpha)$

With the definition of the classes $S_0(\alpha)$ and $T_0(\alpha)$, we have

Theorem 6 If $f(z) \in S_0(\alpha)$ then

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 2\alpha \quad z \in U, \qquad (18)$$

This leads to f(z) is starlike of order 2α in U.

Proof Using the result by Silverman [3] and (3), we see that

$$\sum_{m=2}^{\infty}\{2(m-1)|a_{2m-2}| + (2m-1-2\alpha)|a_{2m-1}|\} \le 1-2\alpha$$

$$\Rightarrow \sum_{m=2}^{\infty}(m-2\alpha)|a_m| \le 1-2\alpha$$

$$\Rightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 2\alpha \qquad z \in E.$$

Theorem 7 If $f(z) \in S_0(\alpha)$, then $f(z) \in R\left(\frac{1}{2(1-\alpha)}\right)$.

Proof Since

$$\sum_{m=2}^{\infty}|a_m| \le 1-\alpha \qquad (0 \le \alpha < 1; z \in E) \qquad (19)$$

Implies $f(z) \in R(\alpha)$, we have

$$f(z) \in S_0(\alpha) \Rightarrow \sum_{m=2}^{\infty}(m-2\alpha)|a_m| \leq 1-2\alpha$$

$$\Rightarrow \sum_{m=2}^{\infty}|a_m| \leq 1-\frac{1}{2(1-\alpha)}$$

$$\Rightarrow f(z) \in R\left(\frac{1}{2(1-\alpha)}\right).$$

Remark Taking $\alpha = 0$ in Theorem 7, we have

$$f(z) \in S_0(\alpha) \Rightarrow f(z) \in R\left(\frac{1}{2}\right)$$

Therefore, noting that $f(z) \in T_0(\alpha)$ if and only if $zf'(z) \in S_0(\alpha)$, we also have

Theorem 8 If $f(z) \in T_0(\alpha)$, then

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 2\alpha \qquad z \in U, \tag{20}$$

That is, $f(z)$ is convex of order 2α in U.

Theorem 9 If $f(z) \in T_0(\alpha)$, then $zf'(z) \in R\left(\frac{1}{2(1-\alpha)}\right)$.

Remark Letting $\alpha = 0$ in Theorem 9, we have

$$f(z) \in T_0(\alpha) \Rightarrow zf'(z) \in R\left(\frac{1}{2}\right).$$

DISTORTION AND GROWTH INEQUALITIES

We derive the distortion and growth inequalities of function in the classes $S_0(\alpha)$ and $T_0(\alpha)$.

Theorem 10 If $f(z) \in S_0(\alpha)$, then

$$|z|-\frac{1-2\alpha}{2}|z|^2-\frac{1-2\alpha}{3-2\alpha}|z|^3 \leq |f(z)| \leq |z|+\frac{1-2\alpha}{2}|z|^2+\frac{1-2\alpha}{3-2\alpha}|z|^3 \tag{21}$$

And

$$1-(1-2\alpha)|z|-\frac{3(1-2\alpha)}{3-2\alpha}|z|^2 \leq |f'(z)| \leq 1+(1-2\alpha)|z|+\frac{3(1-2\alpha)}{3-2\alpha}|z|^3 \tag{22}$$

For $z \in U$. Equalities are attained for functions

$$f(z) = z-\frac{1-2\alpha}{2}z^2-\frac{1-2\alpha}{3-2\alpha}z^3 \tag{23}$$

Or

$$f(z) = z+\frac{1-2\alpha}{2}z^2+\frac{1-2\alpha}{3-2\alpha}z^3 \tag{24}$$

Proof Since we have

$$\sum_{n=2}^{\infty}|a_{2m-2}| \leq \frac{1-2\alpha}{2}$$

And

$$\sum_{m=2}^{\infty}|a_{2m-1}| \leq \frac{1-2\alpha}{3-2\alpha}$$

For $f(z) \in S_0(\alpha)$ we prove

$$|f(z)| \geq |z| - |z|^2 \sum_{m=2}^{\infty} |a_{2n-2}| - |z|^3 \sum_{m=2}^{\infty} |a_{2m-1}|$$
$$\geq |z| - \frac{1-2\alpha}{2}|z|^2 - \frac{1-2\alpha}{3-2\alpha}|z|^3$$

And

$$|f(z)| \leq |z| + |z|^2 \sum_{m=2}^{\infty} |a_{2m-2}| + |z|^3 \sum_{m=2}^{\infty} |a_{2m-1}|$$
$$\leq |z| + \frac{1-2\alpha}{2}|z|^2 + \frac{1-2\alpha}{3-2\alpha}|z|^3.$$

It follows from $f(z) \in S_0(\alpha)$ that

$$\sum_{n=2}^{\infty} 2(m-1)|a_{2m-2}| \leq 1-2\alpha$$

And

$$\frac{3-2\alpha}{3}\sum_{n=2}^{\infty}(2m-1)|a_{2m-1}| \leq \sum_{m=2}^{\infty}(2m-1-2\alpha)|a_{2m-1}| \leq 1-2\alpha,$$

Which implies

$$\sum_{m=2}^{\infty}(2m-1)|a_{2m-1}| \leq \frac{3(1-2\alpha)}{3-2\alpha}.$$

Therefore, we obtain that

$$|f'(z)| \geq 1 - |z| \sum_{m=2}^{\infty} 2(m-1)\, |a_{2m-2}| - |z|^2 \sum_{m=2}^{\infty} (2m-1)\, |a_{2m-1}|$$
$$\geq 1 - (1-2\alpha)|z| - \frac{3(1-2\alpha)}{3-2\alpha}|z|^2$$

And

$$|f'(z)| \leq 1 + |z| \sum_{m=2}^{\infty} 2(m-1)\, |a_{2m-2}| + |z|^2 \sum_{m=2}^{\infty} (2m-1)\, |a_{2m-1}|$$
$$\leq 1 + (1-2\alpha)|z| + \frac{3(1-2\alpha)}{3-2\alpha}|z|^2.$$

This completes the proof of Theorem 10.
Using the same manner in the proof of Theorem 10, we have
Theorem 11 If $f(z) \in T_0(\alpha)$, then

$$|z| - \frac{1-2\alpha}{4}|z|^2 - \frac{1-2\alpha}{3(3-2\alpha)}|z|^3 \leq |f(z)|$$
$$\leq |z| + \frac{1-2\alpha}{4}|z|^2 + \frac{1-2\alpha}{3(3-2\alpha)}|z|^3 \qquad (25)$$

And

$$1 - \frac{1-2\alpha}{2}|z| - \frac{1-2\alpha}{3-2\alpha}|z|^2 \leq |f'(z)|$$
$$\leq 1 + \frac{1-2\alpha}{2}|z| + \frac{1-2\alpha}{3-2\alpha}|z|^3 \qquad (26)$$

For $z \in E$. Equalities are attained for functions

$$f(z) = z - \frac{1-2\alpha}{4}z^2 - \frac{1-2\alpha}{3(3-2\alpha)}z^3 \quad , \qquad (27)$$

or

$$f(z) = z + \frac{1-2\alpha}{4} z^2 + \frac{1-2\alpha}{3(3-2\alpha)} z^3 \qquad (28)$$

THE MAIN RESULTS

The main results in this paper that obtained by studying the specific Sakaguchi functions in the open unit U disk, and certain subcategories of Sakaguchi functions were introduced. Moreover, some properties such as modulus tolerances, some properties of classes $S_0(\alpha)$ and $T_0(\alpha)$, deformation and growth inequality are presented. We mentioned and explained this through the theories that stated in the body of the research.

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